Applications in Algebra and Calculus Assessment Standard 2.1

8. (a) Using the binomial theorem, fully expand $(3x+2)^6$

(3)

OR (b) Expand $(5x - 3)^4$ using the Binomial theorem.

9. Complex numbers are defined as follows: $z_1 = 7 - ni$ and $z_2 = 6 + 4i$.

Express the following in the form $a + ib$:

a) $z_1 z_2$

b) $\dfrac{z_1}{z_2}$.

(3)

Geometry, Proof and Systems of Equations Assessment Standard 3.3

10. Find the modulus and argument of $z = -4\sqrt{3} - 4i$.

Hence express $z = -4\sqrt{3} - 4i$ in polar form.

(3)

11. A complex number *w* has modulus 2 and principal argument $-\dfrac{\pi}{6}$.

(*a*) Plot *w* on an Argand diagram.

(1)

(*b*) Using exact values, express *w* in Cartesian form.

(2)

(8) (a) $(3x+2)^6$

$$
\begin{array}{c}
1\\
1\quad 1\\
1\quad 2\quad 1\\
1\quad 3\quad 3\quad 1\\
1\quad 4\quad 6\quad 4\quad 1\\
1\quad 5\quad 10\quad 10\quad 5\quad 1\\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1
\end{array}
$$

$= (3x)^6 + 6(3x)^5(2) + 15(3x)^4(2)^2 + 20(3x)^3(2)^3 + 15(3x)^2(2)^4 + 6(3x)(2)^5 + 2^6$

$= \underline{\underline{729x^6 + 2916x^5 + 4860x^4 + 4320x^3 + 2160x^2 + 576x + 64}}$

(b) $(5x-3)^4 = (5x+(-3))^4$

$= (5x)^4 + 4(5x)^3(-3) + 6(5x)^2(-3)^2 + 4(5x)(-3)^3 + (-3)^4$

$= \underline{\underline{625x^4 - 1500x^3 + 1350x^2 - 540x + 81}}$

(9) (a) $z_1 z_2 = (7-ni)(6+4i)$

$= 42 + 28i - 6ni - 4ni^2$

$= 42 + 28i - 6ni + 4n \quad$ (since $i^2 = -1$)

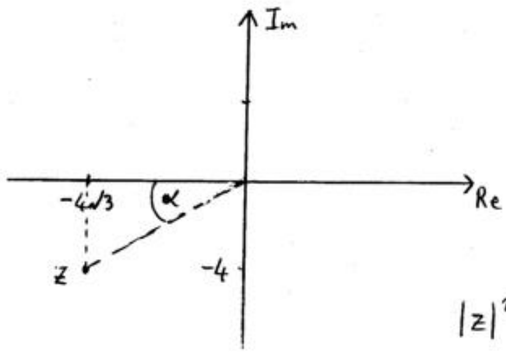
$= \underline{\underline{(42+4n) + (28-6n)i}}$

(b) $\dfrac{z_1}{z_2} = \dfrac{7-ni}{6+4i} = \dfrac{(7-ni)(6-4i)}{(6+4i)(6-4i)} = \dfrac{42-28i-6ni+4ni^2}{6^2-(4i)^2}$

$= \dfrac{(42-4n)-(28+6n)i}{36+16} = \dfrac{42-4n}{52} - \dfrac{28+6n}{52}i$

$= \underline{\underline{\dfrac{21-2n}{26} - \dfrac{14+3n}{26}i}}$

(10)



$\tan\alpha = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$ rad

so $\arg z = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ rad

OR principal $\arg z = -\frac{5\pi}{6}$ rad

$|z|^2 = (-4\sqrt{3})^2 + (-4)^2$

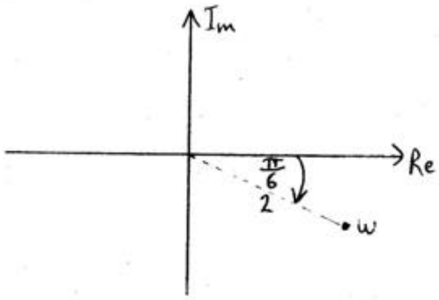
$= 16\times3 + 16$

$= 64$

so $|z| = 8$

$z = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ or $8\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$ OR $8\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$

(11) (a)



(b) $w = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$w = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$w = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$w = \sqrt{3} - i$

Applications in Algebra and Calculus Assessment Standard 2.2

4. An arithmetic sequence is given by: 5, 17, 29, ...

Find

a) the 16th term of the sequence

(2)

b) the sum of the first 14 terms.

(2)

OR Two consecutive terms of an arithmetic sequence are -23 and -31.
The 8th term is -55.

Find

a) the 20th term of the sequence

b) the sum of the first 50 terms.

5. A geometric sequence is given by: 5, -20, 80, ...

a) Find the 15th term of the sequence

(2)

b) Find the sum of the first 12 terms.

(2)

OR A geometric sequence has first term -5 and common ratio 6.

Find

a) the 8th term of the sequence

b) the sum of the first 8 terms.

6. (a) Find the first three non-zero terms of the Maclaurin series for $f(x) = e^{3x}$.

(3)

OR (b) Find the first three non-zero terms of the Maclaurin series for $f(x) = \sin 4x$.

OR (c) Find the first three non-zero terms of the Maclaurin series for $f(x) = \cos 2x$.

4 (a) $u_n = a + (n-1)d$, $a = 5$, $d = 12$, $n = 16$

$u_{16} = 5 + 15 \times 12$

$u_{16} = \underline{\underline{185}}$

(b) $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, so $S_{14} = \frac{14}{2}\left[2 \times 5 + 13 \times 12\right] = 7(10 + 156) = 7 \times 166 = \underline{\underline{1162}}$

Alternative Q4:

(a) $d = -31 - (-23) = -31 + 23 = -8$ $\quad$ $u_8 = -55 = a + 7d = a + 7 \times {}^{-}8$

$-55 = a - 56$

$a = 1$

$u_{20} = 1 + 19 \times {}^{-}8 = 1 - 152 = \underline{\underline{-151.}}$

(b) $S_{50} = \frac{50}{2}\left[2 \times 1 + 49 \times {}^{-}8\right] = 25(2 - 392) = 25 \times -390 = \underline{\underline{-9750}}$

5 (a) $a = 5$ $\quad$ $r = \frac{u_2}{u_1} = \frac{-20}{5} = -4$

$u_n = ar^{n-1}$, so $u_{15} = 5 \times (-4)^{14} = \underline{\underline{1342\ 177\ 280}}$

(b) $S_n = \frac{a(1 - r^n)}{1 - r}$

$S_{12} = \frac{5\left(1 - (-4)^{12}\right)}{1 - (-4)} = 1 - (-4)^{12} = \underline{\underline{-16777215}}$

Alternative Q5:

(a) $a = u_1 = -5$, $\quad r = 6$

$u_n = ar^{n-1}$, so $u_8 = -5 \times 6^7 = \underline{\underline{-1399680}}$

(b) $S_n = \frac{a(r^n - 1)}{r - 1}$, so $S_8 = \frac{-5(6^8 - 1)}{6 - 1} = 1 - 6^8 = \underline{\underline{-1679615}}$

⑥ $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \ldots$

(a) $f(x) = e^{3x} \Rightarrow f(0) = e^0 = 1$

$f'(x) = 3e^{3x} \Rightarrow f'(0) = 3e^0 = 3\times 1 = 3$

$f''(x) = 9e^{3x} \Rightarrow f''(0) = 9e^0 = 9$

$$e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \ldots$$

(b) $f(x) = \sin 4x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = 4\cos 4x \Rightarrow f'(0) = 4\cos 0 = 4\times 1 = 4$

$f''(x) = -16\sin 4x \Rightarrow f''(0) = -16\sin 0 = -16\times 0 = 0$

$f'''(x) = -64\cos 4x \Rightarrow f'''(0) = -64\cos 0 = -64\times 1 = -64$

$f^{iv}(x) = 256\sin 4x \Rightarrow f^{iv}(0) = 256\sin 0 = 0$

$f^{v}(x) = 1024\cos 4x \Rightarrow f^{v}(0) = 1024\cos 0 = 1024$

$$\sin 4x = 4x - \frac{64}{3!}x^3 + \frac{1024}{5!}x^5 \ldots = 4x - \frac{64}{6}x^3 + \frac{1024}{120}x^5 \ldots$$

i.e. $\sin 4x = 4x - \frac{32}{3}x^3 + \frac{128}{15}x^5 \ldots$

⑥ (c) $f(x) = \cos 2x \Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -2\sin 2x \Rightarrow f'(0) = -2\sin 0 = 0$

$f''(x) = -4\cos 2x \Rightarrow f''(0) = -4\cos 0 = -4$

$f'''(x) = 8\sin 2x \Rightarrow f'''(0) = 8\sin 0 = 0$

$f^{iv}(x) = 16\cos 2x \Rightarrow f^{iv}(0) = 16\cos 0 = 16$

$$\cos 2x = 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 \ldots = 1 - \frac{4}{2}x^2 + \frac{16}{24}x^4 \ldots$$

$$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 \ldots$$

⑦ Required to disprove: $a < b$ and $c < d \Rightarrow ac < bd$

Let $a = -4$ and $b = 3$ and $c = -2$ and $d = 1$

$a < b$ and $c < d$

yet $ac = -4\times -2 = 8$ and $bd = 3\times 1 = 3$

so $ac > bd$

i.e. $a < b$ and $c < d \Rightarrow ac < bd$ is false (as shown by the counter-example above)

[Many other numbers could have been chosen.]

Applications in Algebra and Calculus Assessment Standard 2.3

5. (a) Evaluate $\sum_{r=1}^{12} (5r + 2)$ **(3)**

OR Evaluate $\sum_{r=1}^{50} 4r^2$

OR Evaluate $\sum_{r=1}^{40} 3r^3$

5 (a) $\sum_{r=1}^{12} (5r+2) = 5\sum_{r=1}^{12} r + \sum_{r=1}^{12} 2$

$= 5 \times \frac{1}{2} 12(12+1) + 24 \quad \left[\text{since } \sum_{r=1}^{n} r = \frac{1}{2}n(n+1)\right]$

$= 5 \times 6 \times 13 + 24$

$= \underline{\underline{414}}$

(b) $\sum_{r=1}^{50} 4r^2 = 4\sum_{r=1}^{50} r^2 = 4 \times \frac{1}{6} \times 50 \times 51 \times (2 \times 50 + 1) \quad \left[\text{since } \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)\right]$

$= \frac{2}{3} \times 50 \times 51 \times 101$

$= \underline{\underline{171700}}$

(c) $\sum_{r=1}^{40} 3r^3 = 3\sum_{r=1}^{40} r^3 = 3 \times \frac{1}{4} \times 40^2 \times 41^2 \quad \left[\text{since } \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2\right]$

$= \frac{3}{4} \times 1600 \times 1681$

$= \underline{2017200}$