Advanced Higher Mathematics

Complex Numbers

Addition, subtraction and multiplication

EXERCISE 1

1 Given $z_1 = 2 + i$ and $z_2 = 3 + 4i$ calculate each of the following in the form $a + bi$.

a $z_1 + z_2$ **b** z_1z_2 **c** $3z_1$ **d** $2z_2$
e $4z_1 + 3z_2$ **f** z_1^2 **g** z_1^3 **h** $z_1^3z_2$
i $-z_2$ **j** $z_1 - z_2$ **k** $z_2 - z_1$ **l** $z_1^2 - 2z_2$

2 Simplify the following, expressing your answers in the form $a + bi$.

a $(3 + 4i) + (1 + i)$ **b** $(6 - 2i) + (4 + 2i)$ **c** $(1 + i)(1 - i)$
d $(1 + 2i)(1 - 3i)$ **e** $(4 - 3i)^2$ **f** $2(3 - i) - 4(1 + 2i)$
g $3(1 + i) - i(1 + 3i)$ **h** $2i(2 + 3i)(1 - 2i)$ **i** $(3 - i)^2(3 + i)$

3 Solve the following quadratic equations giving the roots in the form $z = a \pm bi$.

a $z^2 + 2z + 2 = 0$ **b** $z^2 + 4z + 13 = 0$ **c** $z^2 - 6z + 13 = 0$
d $2z^2 - 4z + 10 = 0$ **e** $3z^2 - 12z + 15 = 0$ **f** $2z^2 + 12z + 36 = 0$

4 Every quadratic equation can be written in the form

$$z^2 - (\textit{sum of the roots})z + (\textit{product of the roots}) = 0$$

Verify this statement for the equations in question 3.

5 Solve Cardan's problem, namely:
Find two numbers which have a sum of 10 and a product of 40.

6 a Simplify each of the following.
(i) $(3 + i)(3 - i)$ **(ii)** $(2 + 3i)(2 - 3i)$ **(iii)** $(1 + 2i)(1 - 2i)$
b Comment on your answers in each case. Make a conjecture.
c Simplify $(a + ib)(a - ib)$ to prove your conjecture.

7 a $i = i$, $i^2 = -1$, $i^3 = i \times i^2 = -i$, $i^4 = i^2 \times i^2 = 1$
Work out the powers of i up to i^{12}.
b Given that n is an integer, evaluate:
(i) i^{4n-1} **(ii)** i^{4n+1} **(iii)** i^{4n+2} **(iv)** i^{4n} **(v)** i^{4n+3}

8 By equating real and imaginary parts, find a and b in each case.

a $a + bi = (3 + i)^2$ **b** $a + bi = (3 + 2i)^2$ **c** $a + bi = (2 + i)(3 + 4i)$

Division and square roots

EXERCISE 2

1 Calculate the following divisions, expressing your answer in the form $a + ib$ where $a, b \in R$.

a $(8 + 4i) \div (1 + 3i)$ **b** $(8 + i) \div (3 + 2i)$ **c** $(6 + 2i) \div (4 - 2i)$

d $(-1 - 3i) \div (1 - 2i)$ **e** $8 \div (1 + 2i)$ **f** $(6 + i) \div (3 - i)$

2 In each case below, express z^{-1} in the form $a + ib$ where $a, b \in R$.

a $z = i$ **b** $z = 1 - i$ **c** $z = 2 + 2i$

d $z = 3 + i$ **e** $z = 4 - 2i$

3 Simplify:

a $\dfrac{17 - 7i}{5 + i}$ **b** $\dfrac{21 + 9i}{2 + 5i}$ **c** $\dfrac{7 - 3i}{1 + i}$

d $\dfrac{2 - 5i}{1 + i}$ **e** $\dfrac{3 - 2i}{1 + 2i}$ **f** $\dfrac{3}{3 + 4i}$

4 Find a and b in each case so that $(a + ib)^2$ is equal to:

a $5 - 12i$ **b** $15 - 8i$ **c** $-24 - 10i$

5 Calculate:

a $\sqrt{3 - 4i}$ **b** $\sqrt{21 - 20i}$ **c** $\sqrt{-9 + 40i}$

6 **a** If $z = 2 + 3i$ find, in the form $x + iy$:

(i) $\bar{z}$ (ii) $\dfrac{1}{\bar{z}}$ (iii) $\dfrac{z}{\bar{z}}$ (iv) $\dfrac{\bar{z}}{z}$ (v) $\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}$ (vi) $\dfrac{z}{\bar{z}} - \dfrac{\bar{z}}{z}$

b Repeat **a** when $z = a + bi$.

7 **a** Show that $\frac{1}{2}(z + \bar{z}) = \mathcal{R}(z)$.

b Find a similar expression for $\mathcal{I}(z)$.

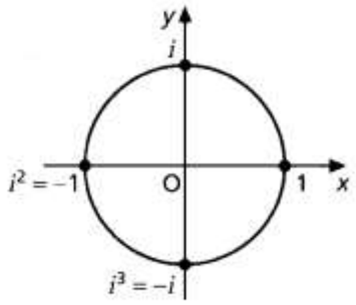
8 Given that $z_1 = a + bi$ and $z_2 = x + iy$,

a find expressions for **(i)** $\bar{z}_1$, **(ii)** $\bar{z}_2$ **(iii)** $\overline{z_1 + z_2}$;

b state the simple relationship between $\bar{z}_1$, $\bar{z}_2$ and $\overline{z_1 + z_2}$;

c identify similar conclusions for **(i)** $\overline{z_1 - z_2}$, **(ii)** $\overline{z_1 \times z_2}$, **(iii)** $\overline{z_1 \div z_2}$.

EXERCISE 3



1 $z = 1 \Rightarrow |z| = 1$ and $\arg z = 0$. Use the diagram to help you make similar statements about

a $z = i$
b $z = -1$
c $z = -i$

2 **a** Draw Argand diagrams to illustrate:
(i) $3 + 4i$ and $3 - 4i$ (ii) $2 + 3i$ and $2 - 3i$ (iii) $5 + i$ and $5 - i$

b Comment on the Argand diagrams of z and $\bar{z}$.

3 For each of the following complex numbers,
(i) plot the number on an Argand diagram,
(ii) find the modulus and argument to three significant figures where appropriate.

a $1 + i$ **b** $2 + 3i$ **c** $3 + 2i$
d 6 **e** $3i$ **f** $-4 - 3i$ (refer to your sketch)
g $-1 + 2i$ **h** $2 - 3i$ **i** $4 - i$

4 For each of the following expressions,
(i) simplify by writing in the form $x + iy$,
(ii) find the modulus and argument.

a $\dfrac{3 + 2i}{1 + 5i}$ **b** $\dfrac{1}{1 + 3i}$ **c** $(2 + 4i)(1 - i)$

5 **a** Find the modulus and argument of each expression.
(i) $10 + 7i$ (ii) $(10 + 7i)^2$ (iii) $(10 + 7i)^3$

b Comment on any connection you see.

6 Find the complex number with:

a $|z| = 2, \arg z = \dfrac{\pi}{6}$ **b** $|z| = 3, \arg z = \dfrac{\pi}{4}$ **c** $|z| = 4, \arg z = \dfrac{\pi}{2}$

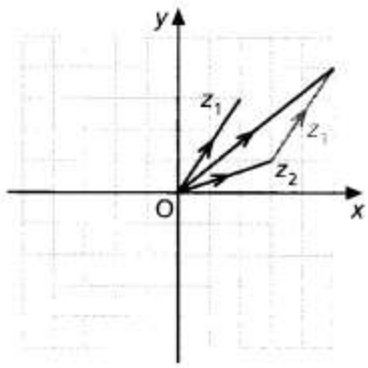
d $|z| = 3, \arg z = \dfrac{\pi}{3}$ **e** $|z| = 2, \arg z = -\dfrac{\pi}{4}$ **f** $|z| = 1, \arg z = -\dfrac{\pi}{6}$

7 Express each of these complex numbers in polar form (give the argument in degrees).

a $1 + i\sqrt{3}$ **b** $\sqrt{2} + i\sqrt{2}$ **c** $-2\sqrt{3} + 2i$
d -1 **e** $3i$ **f** $-4 - i4\sqrt{3}$
g $-2\sqrt{3} - 2i$ **h** $-\sqrt{2} - i\sqrt{2}$ **i** $-1 - i\sqrt{3}$

8 Given that $z_1 = 2 + 3i$ and $z_2 = 3 + i$, the diagram illustrates the sum $z_1 + z_2$, i.e.

$$(2 + 3i) + (3 + i) = 5 + 4i$$



On similar diagrams, illustrate:

a $(2 + 3i) + (3 + 2i)$ **b** $(3 + 3i) + (2 + 2i)$
c $(2 - 3i) + (3 - 2i)$ **d** $(-2 + 3i) + (-3 - 2i)$
e $(2 + 3i) - (3 + 2i)$ [Hint: $= (2 + 3i) + (-3 - 2i)$]
f $(1 + 3i) - (2 + 4i)$ **g** $(-3 + i) - (-1 - 2i)$
h $(-1 - 2i) - (4 - 3i)$ **i** $(-2 - 2i) - (-3 - 3i)$

EXERCISE 4

1 Given that $z = x + iy$, for each of the following, **(i)** find the equation of the locus, **(ii)** draw the locus on an Argand diagram.

a $|z| = 5$ **b** $|z - 3| = 2$ **c** $|z + 1| = 4$ **d** $|z + i| = 3$

e $|z - 2i| = 3$ **f** $|z + 1 + 2i| = 3$ **g** $|2z + 3i| = 5$ **h** $|3z - i| = 5$

i $|3z + 3 - 2i| = 4$ **j** $\arg z = \frac{\pi}{6}$ **k** $\arg z = \frac{\pi}{4}$ **l** $\arg z = \frac{2\pi}{3}$

m $\arg z = 1$ **n** $2 \arg z = \frac{\pi}{4}$ **o** $\arg z = -\frac{\pi}{3}$

2 Explore the loci of the form:

a $|z - a| = b$ **b** $|z - ai| = b$ **c** $|z - ai - b| = c$

3 Given that $z = x + iy$, find the equation of the locus in each case.

a $|z - 1| = |z - i|$ **b** $|z - 2| = |z - i|$ **c** $|z - 3| = |z - 2i|$

d $|z - a| = |z - bi|$

4 Sketch the following loci, given that $z = x + iy$.

a $|z| \le 3$ **b** $|z - 3| \le 2$ **c** $|z + 2| \ge 5$

Historical note

When two whole numbers are added, the result is another whole number. The whole numbers are said to be *closed* under addition.

$$a, b \in W \Rightarrow a + b \in W$$

However, when two whole numbers are subtracted, the result is not always another whole number. Consider $2 - 3 = -1$: not a whole number. The whole numbers are not closed under subtraction.

Diophantus (around AD275) called equations which produced negative numbers *absurd*. By the sixteenth century, negative numbers were perfectly acceptable, and the set of whole numbers was extended, by the inclusion of negative numbers, to the set of integers.

Consider 2 and 3 as integers, $2 - 3 = -1$: an integer.
The integers are closed under subtraction.

$$a, b \in Z \Rightarrow a - b \in Z$$

Consider closure under division and you will see the need for the set of rational numbers. Consider closure when taking the square root and you will see the need for the set of real numbers.

Jerome Cardan

In 1545 Jerome Cardan tried to find the solution to the problem: *What two numbers have a sum of 10 and a product of 40?*
His solution involved what he termed as *fictitious* numbers, what we now call *complex numbers*.

In 1637, Descartes used the expressions *real* and *imaginary* in this context and in 1748 Euler used the letter i to stand for the root of the equation $x^2 = -1$.

If we wish to work with $\sqrt{-1}$ we shall need to extend the set of real numbers.

EXERCISE 6

1 For each of the following complex numbers z,

- express it in polar form,
- find each of the required powers in polar form, bringing the argument into the range $(-\pi, \pi]$,
- finally express your answers in the form $a + ib$.

a Given $z = 2\sqrt{3} + 2i$ find **(i)** z^2 **(ii)** z^5 **(iii)** z^{10}

b Given $z = \sqrt{3} - i$ find **(i)** z^3 **(ii)** z^4 **(iii)** z^8

c Given $z = 1 + i$ find **(i)** z^3 **(ii)** z^6 **(iii)** z^{12}

2 Simplify, giving your answers correct to three significant figures:

a $\left[3\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right]^3$

b $\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^8$

c $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^2$

d $\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^3\left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)^4$

3 In each of the following, work to three decimal places where necessary then round the components of your final answer to the nearest whole number.

a Given $z = 2 + 3i$, calculate z^3.

b Given $z = -1 + 4i$, calculate z^5.

c Given $z = -2 - 3i$, calculate z^4.

d Given $z = 2 - 2i$, calculate z^7.

4 The argument can be given in degrees. The same laws apply and the range of the argument is $(-180°, 180°]$. Simplify the following:

a $(\cos 10° + i\sin 10°)(\cos 30° + i\sin 30°)$

b $(\cos 50° + i\sin 50°)(\cos 145° + i\sin 145°)$

c $(\cos 25° + i\sin 25°)(\cos 20° - i\sin 20°)$

d $(\cos 150° - i\sin 150°)(\cos 40° - i\sin 40°)$

e $(\cos 30° + i\sin 30°) \div (\cos 10° + i\sin 10°)$

f $(\cos 4° + i\sin 4°) \div (\cos 10° + i\sin 10°)$

g $(\cos 20° + i\sin 20°)^3(\cos 30° + i\sin 30°)^2$

h $(\cos 125° - i\sin 125°)^4(\cos 15° - i\sin 15°)^3$

i $\dfrac{(\cos 40° + i\sin 40°)^3}{(\cos 10° + i\sin 10°)^2}$

j $\dfrac{(\cos 6° + i\sin 6°)^5}{(\cos 3° + i\sin 3°)^2}$

k $\dfrac{(\cos 25° + i\sin 25°)^4}{(\cos 7° + i\sin 7°)(\cos 3° + i\sin 3°)}$

l $\dfrac{(\cos 32° + i\sin 32°)^4}{(\cos 16° + i\sin 16°)^3(\cos 4° - i\sin 4°)^2}$

5 a Expand $(\cos\theta + i\sin\theta)^2$ (i) using the binomial theorem,
(ii) using De Moivre's theorem.

b (i) By equating the *real* parts, express $\cos 2\theta$ in terms of $\sin\theta$ and $\cos\theta$.
(ii) By equating the *imaginary* parts, express $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$.

6 **a** Expand $(\cos\theta + i\sin\theta)^3$ (i) using the binomial theorem,
(ii) using De Moivre's theorem.

b (i) By equating the *real* parts, express $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.
(ii) Use the identity $\sin^2\theta + \cos^2\theta = 1$ to help you express $\cos 3\theta$ in terms of $\cos\theta$ only.

c Express $\sin 3\theta$ in terms of $\sin\theta$.

d Hence express $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$.

7 **a** By considering the expansion of $(\cos\theta + i\sin\theta)^4$, express:
(i) $\cos 4\theta$ in terms of $\cos\theta$
(ii) $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$
(iii) $\cos^4\theta$ in terms of $\cos\theta$ and $\cos 4\theta$

b Express:
(i) $\cos 5\theta$ in terms of $\cos\theta$
(ii) $\sin 5\theta$ in terms of $\sin\theta$
(iii) $\cos^5\theta$ in terms of $\cos\theta$ and $\cos 5\theta$

8 $z_1 = \cos\dfrac{11\pi}{6} - i\sin\dfrac{11\pi}{6}$, $z_2 = \cos\dfrac{5\pi}{6} - i\sin\dfrac{5\pi}{6}$, $z_3 = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}$,
$z_4 = \cos\dfrac{7\pi}{6} + i\sin\dfrac{7\pi}{6}$.

a Find expressions for (i) z_1^2, (ii) z_2^2, (iii) z_3^2 (iv) z_4^2.

b (i) Reduce each argument so that it lies in the range $(-\pi, \pi]$.
(ii) How many distinct answers are obtained?

c (i) Reduce the arguments of z_1, z_2, z_3 and z_4.
(ii) If asked for $z \in C$, such that $z^2 = \left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$, what would be a complete answer?
(iii) Illustrate the set of solutions on an Argand diagram.

9 $z_1 = \cos\dfrac{13\pi}{12} - i\sin\dfrac{13\pi}{12}$, $z_2 = \cos\dfrac{5\pi}{12} - i\sin\dfrac{5\pi}{12}$, $z_3 = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$,
$z_4 = \cos\dfrac{11\pi}{12} + i\sin\dfrac{11\pi}{12}$, $z_5 = \cos\dfrac{19\pi}{12} + i\sin\dfrac{19\pi}{12}$.

a Verify in each case that $z^3 = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}$.

b (i) Reduce the arguments of z_1, z_2, z_3, z_4 and z_5 to lie in the range $(-\pi, \pi]$.
(ii) If asked for $z \in C$, such that $z^3 = \left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$, what would be a complete answer?
(iii) Illustrate the set of solutions on an Argand diagram.

Example Given $f(z) = z^4 - 6z^3 + 18z^2 - 30z + 25$,

a show that $z = 1 + 2i$ is a root of the equation $f(z) = 0$,

b hence find the other roots of the polynomial.

a $$\begin{aligned} f(1+2i) &= (1+2i)^4 - 6(1+2i)^3 + 18(1+2i)^2 - 30(1+2i) + 25 \\ &= (-7-24i) - 6(-11-2i) + 18(-3+4i) - 30(1+2i) + 25 \\ &= -7 - 24i + 66 + 12i - 54 + 72i - 30 - 60i + 25 \\ &= 0 \end{aligned}$$

Thus $z = 1 + 2i$ is a root.

b If $z = 1 + 2i$ is a root then, since $f(z)$ has real coefficients, the conjugate, $1 - 2i$, is also a root.
Thus $(z - (1 + 2i))$ and $(z - (1 - 2i))$ are complex factors of $f(z)$. These multiply to give the real quadratic factor $(z - 1 - 2i)(z - 1 + 2i) = z^2 - 2z + 5$.

By division

$$\begin{array}{r|l} & z^2 - 4z + 5 \\ \hline z^2 - 2z + 5 & z^4 - 6z^3 + 18z^2 - 30z + 25 \\ & \underline{z^4 - 2z^3 + 5z^2} \\ & -4z^3 + 13z^2 - 30z + 25 \\ & \underline{-4z^3 + 8z^2 - 20z} \\ & 5z^2 - 10z + 25 \\ & \underline{5z^2 - 10z + 25} \\ & 0 \end{array}$$

We find that the complementary real factor is $z^2 - 4z + 5$.

Equating this to zero and using the quadratic formula

$$z = \frac{4 \pm \sqrt{(16 - 20)}}{2} = 2 \pm i$$

We now have all four roots, which are $1 + 2i$, $1 - 2i$, $2 + i$ and $2 - i$.

Complex polynomials

EXERCISE 8

1 Use the quadratic formula to find the two complex roots of each of the following equations.

a $z^2 - 2z + 10 = 0$ **b** $z^2 - 4z + 5 = 0$ **c** $z^2 - 6z + 25 = 0$

d $4z^2 - 16z + 17 = 0$ **e** $2z^2 + 2z + 1 = 0$ **f** $5z^2 + 4z + 8 = 0$

2 For each cubic equation, a real root has been identified. Find the remaining two complex roots.

a $z^3 + 2z^2 + z + 2 = 0$; $z = -2$ **b** $z^3 - z^2 - z - 15 = 0$; $z = 3$

c $z^3 + 6z^2 + 37z + 58 = 0$; $z = -2$ **d** $z^3 - 11z - 20 = 0$; $z = 4$

e $4z^3 + 8z^2 + 14z + 10 = 0$; $z = -1$ **f** $2z^3 + 11z^2 + 20z - 13 = 0$; $z = \frac{1}{2}$

3 For each cubic equation, a factor has been given. Find the roots of the equation.

a $z^3 + z^2 - 7z - 15 = 0;\quad z - 3$

b $z^3 - 6z^2 + 13z - 20 = 0;\quad z - 4$

c $2z^3 + 5z^2 + 8z + 20 = 0;\quad z - 2i$

d $2z^3 + 7z^2 + 26z + 30 = 0;\quad z + 1 + 3i$

e $4z^3 - 4z^2 + 9z - 9 = 0;\quad 2z - 3i$

f $2z^3 + 13z^2 + 46z + 65 = 0;\quad z + 2 - 3i$

4 Given $f(z) = z^4 - z^3 - 2z^2 + 6z - 4$ and that $z = 1 + i$ is a root of the equation $f(z) = 0$, find the real factors of the polynomial.

5 $1 + 2i$ is a root of the equation $z^4 - 5z^3 + 13z^2 - 19z + 10$. Find the real factors of the polynomial.

6 In each of the following,

(i) show that the given complex number is a zero of the given polynomial,

(ii) find all the remaining roots.

a $z = 2 + i;\quad f(z) = z^4 - 2z^3 - z^2 + 2z + 10$

b $z = 3 + 2i;\quad f(z) = z^4 - 8z^3 + 30z^2 - 56z + 65$

c $z = 1 + 3i;\quad f(z) = 2z^4 - 3z^3 + 17z^2 + 12z - 10$

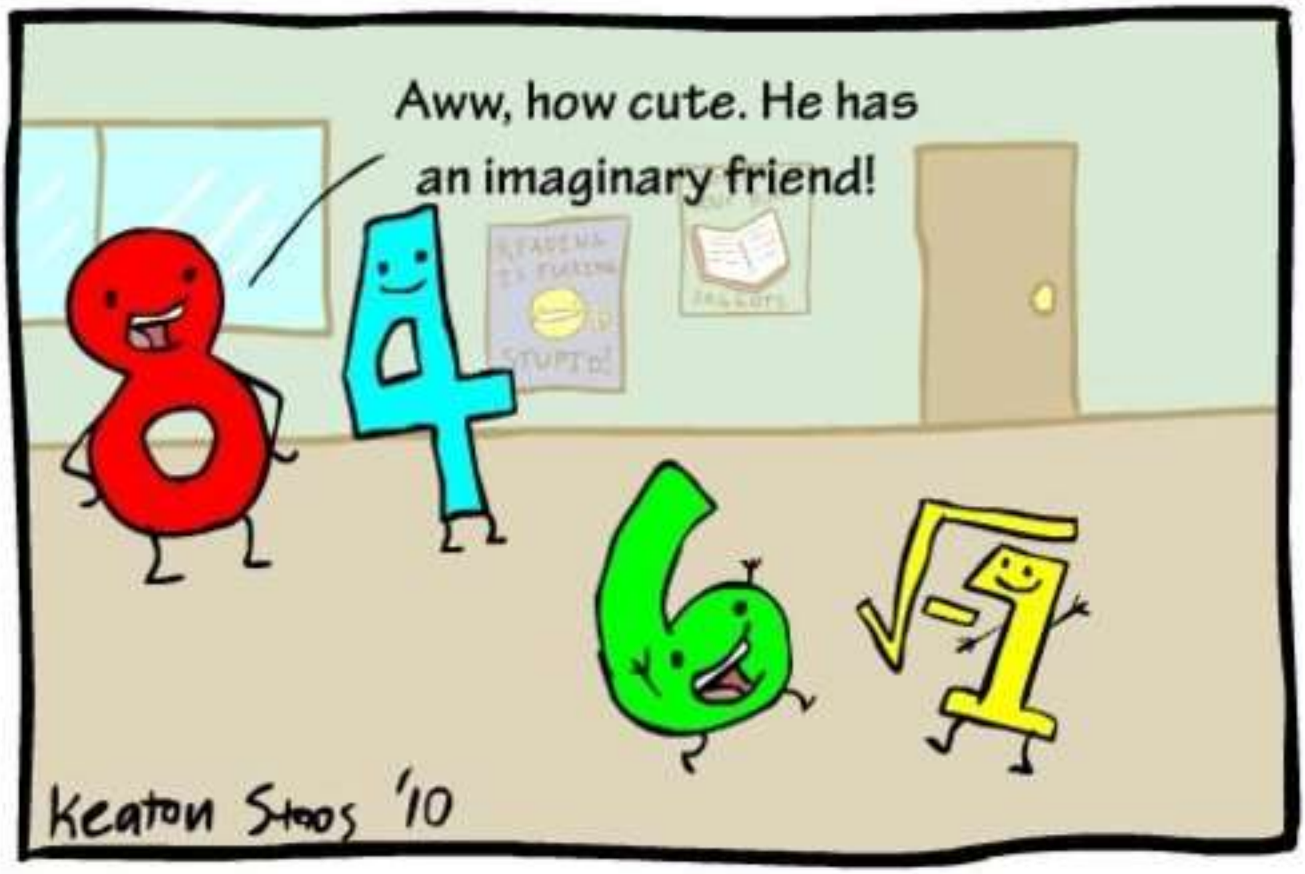
7 In each case the given complex number is a zero of the given polynomial. Find all the roots.

a $z = 4 + i;\quad f(z) = z^4 - 8z^3 + 13z^2 + 32z - 68$

b $z = 2 - 3i;\quad f(z) = 6z^4 - 31z^3 + 108z^2 - 99z + 26$

c $z = 4 + 2i;\quad f(z) = 2z^4 - 25z^3 + 107z^2 - 140z - 100$

8 $z^2 + 4z + 5$ is a factor of $z^5 + 7z^4 + 21z^3 + 33z^2 + 28z + 10$. Find all the roots of the equation $z^5 + 7z^4 + 21z^3 + 33z^2 + 28z + 10 = 0$.



CHAPTER 4

Exercise 1 (page 90)

1 **a** $5 + 5i$ **b** $2 + 11i$ **c** $6 + 3i$
d $6 + 8i$ **e** $17 + 16i$ **f** $3 + 4i$
g $2 + 11i$ **h** $-38 + 41i$ **i** $-3 - 4i$
j $-1 - 3i$ **k** $1 + 3i$ **l** $-3 - i$

2 **a** $4 + 5i$ **b** 10 **c** 2
d $7 - i$ **e** $7 - 24i$ **f** $2 - 10i$
g $6 + 2i$ **h** $2 + 16i$ **i** $30 - 10i$

3 **a** $z = -1 \pm i$ **b** $-2 \pm 3i$ **c** $3 \pm 2i$
d $1 \pm 2i$ **e** $2 \pm i$ **f** $-3 \pm 3i$

4 proofs

5 $5 \pm \sqrt{15}i$

6 **a** (i) 10 (ii) 13 (iii) 5
b all real **c** $a^2 + b^2$

7 **a** $i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, 1$
b (i) $-i$ (ii) i (iii) -1 (iv) 1 (v) $-i$

8 **a** $a = 8, b = 6$ **b** $a = 5, b = 12$
c $a = 2, b = 11$

Exercise 2 (page 91)

1 **a** $2 - 2i$ **b** $2 - i$ **c** $1 + i$
d $1 - i$ **e** $1.6 - 3.2i$
f $1.7 + 0.9i$

2 **a** $-i$ **b** $\frac{1}{2} + \frac{1}{2}i$ **c** $\frac{1}{4} - \frac{1}{4}i$
d $\frac{3}{10} - \frac{1}{10}i$ **e** $\frac{1}{5} + \frac{1}{10}i$

3 **a** $3 - 2i$ **b** $3 - 3i$ **c** $2 - 5i$
d $-\frac{3}{2} - \frac{7}{2}i$ **e** $-0.2 - 1.6i$
f $0.36 - 0.48i$

4 **a** 3, −2 or −3, 2 **b** 4, −1 or −4, 1
c 1, −5 or −1, 5

5 **a** $2 - i$ or $-2 + i$ **b** $5 - 2i$ or $-5 + 2i$
c $4 + 5i$ or $-4 - 5i$

6 **a** (i) $2 - 3i$ (ii) $\frac{2}{13} + \frac{3}{13}i$
(iii) $-\frac{5}{13} + \frac{12}{13}i$ (iv) $-\frac{5}{13} - \frac{12}{13}i$
(v) $-\frac{10}{13}$ (vi) $\frac{24}{13}i$

b (i) $a - ib$ (ii) $\frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2}i$
(iii) $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$
(iv) $\frac{a^2 - b^2}{a^2 + b^2} - \frac{2ab}{a^2 + b^2}i$
(v) $\frac{2(a^2 - b^2)}{a^2 + b^2}$ (vi) $\frac{4ab}{a^2 + b^2}i$

7 **b** $\mathcal{I}(z) = \frac{1}{2i}(z + \bar{z})$

8 **a** (i) $a - bi$ (ii) $x - yi$
(iii) $(a + x) - (b + y)i$
b $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$
c (i) $\bar{z}_1 - \bar{z}_2$ (ii) $\bar{z}_1 \times \bar{z}_2$ (iii) $\bar{z}_1 \div \bar{z}_2$

Exercise 3 (page 94)

1 **a** $|z| = 1$; $\arg(z) = \frac{\pi}{2}$
b $|z| = 1$; $\arg(z) = \pi$
c $|z| = 1$; $\arg(z) = -\frac{\pi}{2}$

2 **a** (i) (3, 4), (3, −4) (ii) (2, 3), (2, −3)
(iii) (5, 1), (5, − 1)
b reflection in x-axis

3 **a** (i) (1, 1) (ii) 1.41, 45°
b (i) (2, 3) (ii) 3.61, 56.3°
c (i) (3, 2) (ii) 3.61, 33.7°
d (i) (6, 0) (ii) 6, 0°
e (i) (0, 3) (ii) 3, 90°
f (i) (−4, −3) (ii) 5, −143°
g (i) (−1, 2) (ii) 2.24, 117°
h (i) (2, −3) (ii) 3.61, −56.3°
i (i) (4, −1) (ii) 4.12, −14.0°

4 **a** (i) $\frac{1}{2} - \frac{1}{2}i$ (ii) 0.707, −45°
b (i) $\frac{1}{10} - \frac{3}{10}i$ (ii) 0.316, −71.6°
c (i) $6 + 2i$ (ii) 6.32, 18.4°

5 **a** (i) 12.2, 35° (ii) 149, 70°
(iii) 1819, 105°
b Raise a number to the nth power and you raise the modulus by the nth power and increase the argument by a factor of n.

6 **a** $\sqrt{3} + i$ **b** $\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$ **c** $4i$
d $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ **e** $\sqrt{2} - \sqrt{2}i$ **f** $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

7 **a** $2(\cos 60° + i \sin 60°)$
b $2(\cos 45° + i \sin 45°)$
c $4(\cos 150° + i \sin 150°)$
d $(\cos 180° + i \sin 180°)$
e $3(\cos 90° + i \sin 90°)$
f $8(\cos 120° - i \sin 120°)$
g $4(\cos 150° - i \sin 150°)$
h $2(\cos 135° - i \sin 135°)$
i $2(\cos 120° - i \sin 120°)$

8 diagrams to illustrate
a (5, 5) **b** (5, 5) **c** (5, −5)
d (−5, 1) **e** (−1, 1) **f** (−1, −1)
g (−2, 3) **h** (−5, 1) **i** (1, 1)

Exercise 4 (page 96)

1 **a** $x^2 + y^2 = 25$ **b** $(x - 3)^2 + y^2 = 4$
c $(x + 1)^2 + y^2 = 16$ **d** $x^2 + (y + 1)^2 = 9$
e $x^2 + (y - 2)^2 = 9$
f $(x + 1)^2 + (y + 2)^2 = 9$
g $(2x)^2 + (2y + 3)^2 = 25$
h $(3x)^2 + (3y - 1)^2 = 25$
i $(3x + 3)^2 + (3y - 2)^2 = 16$
j $y = \frac{x}{\sqrt{3}}$ **k** $y = x$

l $y = -\sqrt{3}x$ **m** $y = x \tan 1$

n $y = \tan \dfrac{\pi}{8}x$ **o** $y = -\sqrt{3}x$

2 **a** circle, centre $(a, 0)$, radius b

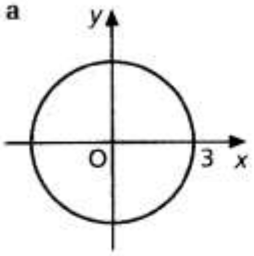
b circle, centre $(0, a)$, radius b

c circle, centre (b, a), radius c

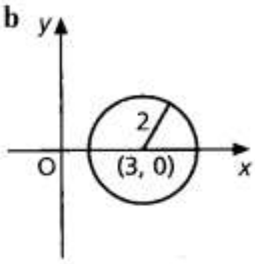
3 **a** $x = y$ **b** $y = 2x - \frac{3}{2}$

c $y = \frac{3}{2}x - \frac{5}{4}$ **d** $\dfrac{a}{b}x - \dfrac{a^2 - b^2}{2b} = y$

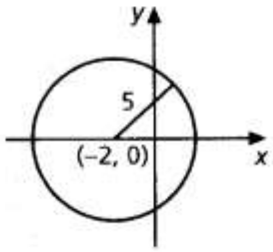
4 **a**



b



c



Note. In order to save space, in the answers for the rest of this chapter an abbreviation cis is used as follows: $a \text{ cis } x = a(\cos x + i \sin x)$, e.g.

$$3 \text{ cis } \frac{\pi}{4} = 3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right),$$

$$4 \text{ cis}(-20°) = 4(\cos(-20°) + i \sin(-20°)) = 4(\cos 20° - i \sin 20°).$$

Exercise 5 (page 98)

1 **a** $12 \text{ cis } \dfrac{5\pi}{6}$ **b** $10 \text{ cis } \dfrac{5\pi}{12}$

c $8 \text{ cis } 0 = 8$ **d** $2 \text{ cis } \left(-\dfrac{5\pi}{6}\right)$

e $10 \text{ cis } \left(-\dfrac{\pi}{30}\right)$ **f** $2 \text{ cis } \dfrac{\pi}{6}$

g $\dfrac{5}{2} \text{ cis } \dfrac{\pi}{8}$ **h** $3 \text{ cis } \dfrac{2\pi}{3}$

i $4 \text{ cis } \dfrac{\pi}{7}$

2 **a** $z_1 = 5 \text{ cis } (0.927)$,
$z_2 = \sqrt{2} \text{ cis } (0.785)$,
$z_1z_2 = 7.07 \text{ cis } (1.712)$,
$\dfrac{z_1}{z_2} = 3.54 \text{ cis } (0.142)$

b $z_1 = 3.61 \text{ cis } (0.983)$,
$z_2 = 3.16 \text{ cis } (-0.321)$,
$z_1z_2 = 11.4 \text{ cis } (0.661)$,
$\dfrac{z_1}{z_2} = 1.14 \text{ cis } (1.30)$

c $z_1 = 1.414 \text{ cis } (-0.785)$,
$z_2 = 1.414 \text{ cis } (-2.35)$,
$z_1z_2 = 2 \text{ cis } (\pi)$,
$\dfrac{z_1}{z_2} = 1 \text{ cis } \left(\dfrac{\pi}{2}\right)$

d $z_1 = 4.12 \text{ cis } (2.90)$,
$z_2 = 2.83 \text{ cis } (2.36)$,
$z_1z_2 = 11.7 \text{ cis } (-1.03)$,
$\dfrac{z_1}{z_2} = 1.46 \text{ cis } (0.540)$

3 **a** $r^2 \text{ cis } \dfrac{2\pi}{3}$ **b** $r^3 \text{ cis } \pi$

c $r^4 \text{ cis } \dfrac{-2\pi}{3}$ **d** $r^5 \text{ cis } \dfrac{-\pi}{3}$

e $r^6 \text{ cis } 0$ **f** $r^7 \text{ cis } \dfrac{\pi}{3}$

4 **(i)** **a** $r^2 \text{ cis } \pi$ **b** $r^3 \text{ cis } \dfrac{-\pi}{2}$

c $r^4 \text{ cis } 0$ **d** $r^5 \text{ cis } \dfrac{\pi}{2}$

e $r^6 \text{ cis } \pi$ **f** $r^7 \text{ cis } \dfrac{-\pi}{2}$

(ii) **a** $r^2 \text{ cis } \dfrac{-2\pi}{3}$ **b** $r^3 \text{ cis } 0$

c $r^4 \text{ cis } \dfrac{2\pi}{3}$ **d** $r^5 \text{ cis } \dfrac{-2\pi}{3}$

e $r^6 \text{ cis } 0$ **f** $r^7 \text{ cis } \dfrac{2\pi}{3}$

(iii) **a** $r^2 \text{ cis } \dfrac{-2\pi}{4}$ **b** $r^3 \text{ cis } \dfrac{\pi}{4}$

c $r^4 \text{ cis } \pi$ **d** $r^5 \text{ cis } \dfrac{-\pi}{4}$

e $r^6 \text{ cis } \dfrac{2\pi}{4}$ **f** $r^7 \text{ cis } \dfrac{-3\pi}{4}$

(iv) **a** $r^2 \text{ cis } 2\theta$ **b** $r^3 \text{ cis } 3\theta$
c $r^4 \text{ cis } 4\theta$ **d** $r^5 \text{ cis } 5\theta$
e $r^6 \text{ cis } 6\theta$ **f** $r^7 \text{ cis } 7\theta$

5 **a** $-527 - 336i$

b **(i)** $5 \text{ cis } (0.927)$
(ii) $625 \text{ cis } (-2.575)$
(iii) $-527 - 336i$

c method **b**

Exercise 6 (page 101)

1 **a** **(i)** $8 + 13.9i$ **(ii)** $-887 + 512i$
(iii) $524\,288 - 908\,093i$

b **(i)** $-8i$ **(ii)** $-8 - 13.9i$
(iii) $-128 + 221.7i$

c **(i)** $-2 + 2i$ **(ii)** $-8i$
(iii) -64

2 **a** $-8.34 + 25.6i$ **b** $-128 - 222i$

c 1 **d** $-0.223 + 0.975i$

3 a $-46 + 9i$ b $-1121 + 404i$
c $-119 - 120i$ d $1024 + 1024i$

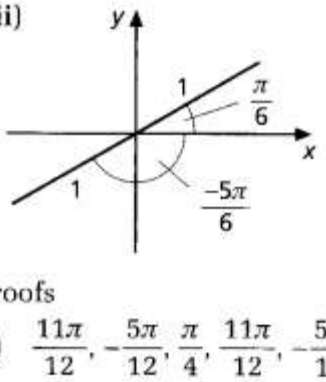
4 a cis 40° b cis (−165)°
c cis 5° d cis 170°
e cis 20° f cis (−6)°
g cis 120° h cis 175°
i cis 100° j cis 24°
k cis 90° l cis 88°

5 a (i) $\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$
(ii) $\cos 2\theta + i\sin 2\theta$
b (i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
(ii) $\sin 2\theta = 2\sin\theta\cos\theta$

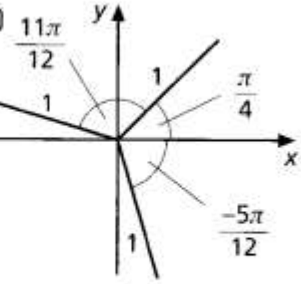
6 a (i) $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$
(ii) $\cos 3\theta + i\sin 3\theta$
b (i) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
c $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$
d $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

7 a (i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
(ii) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$
(iii) $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 8\cos^2\theta - 1)$
b (i) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$
(ii) $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$
(iii) $\cos^5\theta = \frac{1}{16}(\cos 5\theta - 5\cos\theta + 20\cos^3\theta)$

8 a (i) $\text{cis}\left(-\frac{11\pi}{3}\right)$ (ii) $\text{cis}\left(-\frac{5\pi}{3}\right)$
(iii) $\text{cis}\left(\frac{\pi}{3}\right)$ (iv) $\text{cis}\left(\frac{7\pi}{3}\right)$
b (i) $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$
(ii) one distinct answer
c (i) $\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, -\frac{5\pi}{6}$
(ii) $\text{cis}\left(\frac{\pi}{6}\right), \text{cis}\left(-\frac{5\pi}{6}\right)$
(iii)



9 a proofs
b (i) $\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{4}, \frac{11\pi}{12}, -\frac{5\pi}{12}$
(ii) $\text{cis}\left(\frac{11\pi}{12}\right), \text{cis}\left(-\frac{5\pi}{12}\right), \text{cis}\left(\frac{\pi}{4}\right)$
(iii)



Exercise 7 (page 106)

1 a $2\ \text{cis}\left(\frac{1}{3}\left(\frac{\pi}{4} + 2k\pi\right)\right), k = 0, 1, 2$
b $\text{cis}\left(\frac{1}{4}\left(\frac{\pi}{5} + 2k\pi\right)\right), k = 0, 1, 2, 3$
c $2\ \text{cis}\left(\frac{1}{5}\left(\frac{\pi}{7} + 2k\pi\right)\right), k = 0, \ldots, 4$
d $4\ \text{cis}\left(\frac{1}{3}\left(\frac{2\pi}{3} + 2k\pi\right)\right), k = 0, 1, 2$
e $2\ \text{cis}\left(\frac{1}{5}\left(-\frac{\pi}{7} + 2k\pi\right)\right), k = 0, \ldots, 4$
f $4\ \text{cis}\left(\frac{1}{3}\left(-\frac{2\pi}{3} + 2k\pi\right)\right), k = 0, 1, 2$
g $2^{\frac{3}{8}}\ \text{cis}\left(\frac{1}{4}\left(-\frac{3\pi}{4} + 2k\pi\right)\right), k = 0, 1, 2, 3$
h $6^{\frac{1}{5}}\ \text{cis}\left(\frac{1}{5}\left(\frac{2\pi}{3} + 2k\pi\right)\right), k = 0, \ldots, 4$
i $8^{\frac{1}{8}}\ \text{cis}\left(\frac{1}{4}\left(\frac{3\pi}{4} + 2k\pi\right)\right), k = 0, 1, 2, 3$
j $6^{\frac{1}{5}}\ \text{cis}\left(\frac{1}{5}\left(-\frac{2\pi}{3} + 2k\pi\right)\right), k = 0, \ldots, 4$

2 a $\text{cis}\left(\frac{1}{3}(2k\pi)\right)$ b $\text{cis}\left(\frac{1}{4}(2k\pi)\right)$
c $\text{cis}\left(\frac{1}{6}(2k\pi)\right)$ d $3\ \text{cis}\left(\frac{1}{4}(2k\pi)\right)$
e (i) $\text{cis}\left(\frac{1}{5}(\pi + 2k\pi)\right)$ (ii) $\left(\frac{1}{5}\left(\frac{\pi}{2} + 2k\pi\right)\right)$
(iii) $\text{cis}\left(\frac{1}{5}\left(-\frac{\pi}{2} + 2k\pi\right)\right)$
f (i) $4\ \text{cis}\left(\frac{1}{3}(\pi + 2k\pi)\right)$
(ii) $5\ \text{cis}\left(\frac{1}{4}\left(\frac{\pi}{2} + 2k\pi\right)\right)$
(iii) $\frac{1}{2}\ \text{cis}\left(\frac{1}{5}\left(-\frac{\pi}{2} + 2k\pi\right)\right)$

Exercise 8 (page 108)

1 a $1 \pm 3i$ b $2 \pm i$ c $3 \pm 4i$
d $2 \pm 0.5i$ e $-\frac{1}{2} \pm \frac{1}{2}i$
f $-0.4 \pm 1.2i$

2 a $\pm i$ b $-1 \pm 2i$ c $-2 \pm 5i$
d $-2 \pm i$ e $-0.5 \pm 1.5i$ f $-3 \pm 2i$

3 **a** $3, -2 \pm i$ **b** $4, 1 \pm 2i$
c $-2.5, \pm 2i$ **d** $-1.5, -1 \pm 3i$
e $\pm 1.5i, 1$ **f** $-2 \pm 3i, -2.5$

4 $(z + 2)(z - 1), z^2 - 2z + 2$

5 $(z - 2)(z - 1), z^2 - 2z + 5$

6 **a** $2 - i, -1 \pm i,$ **b** $3 - 2i, 1 \pm 2i$
c $1 - 3i, -1, 0.5$

7 **a** $4 \pm i, \pm 2$ **b** $(2 \pm 3i), \frac{2}{3}, \frac{1}{2}$
c $(4 \pm 2i), 5, -\frac{1}{2}$

8 $-2 \pm i, -1 \pm i, -1$

Review (page 110)

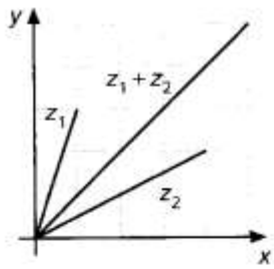
1 **a** $8 - 8i$ **b** $2 - 16i$
c $63 - 16i$ **d** $5 + 12i$
e $-1.32 - 2.24i$ **f** $3 - 2i$ or $-3 + 2i$
g 3

2 $a = 2, b = 1$

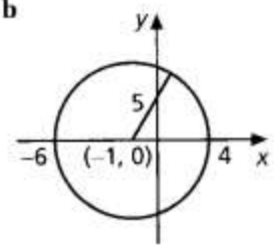
3 $13(\cos 67.4° + i \sin 67.4°)$

4 $3.54, -81.9°$ (3 s.f.)

5



6 **a** $(x + 1)^2 + y^2 = 25$
b



7 **a** $6 \text{ cis } \frac{\pi}{2}$ **b** $\frac{2}{3} \text{ cis } \frac{\pi}{6}$ **c** $8 \text{ cis } \pi$

8 **a** $256 \text{ cis } (-120°)$ **b** $-128 - 221.7i$

9 **a** (i) $\cos^5\theta + 5i\cos^4\theta\sin\theta$
$- 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$
$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$
(ii) $\cos 5\theta + i \sin 5\theta$
b $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

10 **a** $z = 5 \text{ cis } \left(\frac{1}{3}\left(\frac{\pi}{4} + 2k\pi\right)\right), k = 0, 1, 2$

11 $\text{cis } \left(\frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

12 $(z + 2), (z - 1), z^2 - 2z + 17$

13 $2 \pm 3i, 1 \pm 2i$