Outcome 4 – Properties of Functions

Find the vertical asymptote(s) of a rational function. (a function of the form $\frac{P(x)}{Q(x)}$)

Asymptotes

An **Asymptote** is a straight line to which a curve approaches more and more closely as x becomes larger or smaller, or approaches a certain value.

Use the computer or a graphics calculator to obtain the graphs of the following rational functions.

Show the asymptotes clearly and write down their equations.

1. $f(x) = \frac{1}{x}$
2. $f(x) = \frac{1}{x-3}$
3. $f(x) = \frac{1}{x+2}$
4. $f(x) = \frac{2x+3}{3-x}$
5. $f(x) = \frac{1}{(x-1)(x+3)}$
6. $f(x) = \frac{x}{x-1}$
7. $f(x) = \frac{x}{(x+2)(x-3)}$
8. $f(x) = \frac{x^2}{(x+2)(x-3)}$
9. $f(x) = \frac{x-1}{x^2}$
10. $f(x) = \frac{x^2}{x-1}$
11. $f(x) = \frac{x^3}{x^2-1}$
12. $f(x) = \frac{1}{x^2+1}$

When using a computer or a graphics calculator, it is not always possible to be sure that you have identified the equation of the asymptote correctly.
A little algebra will identify the equations.
You will have seen that there are **vertical** and **non - vertical** asymptotes.

Vertical asymptotes are found from the zeros of the denominator ie. are in the form $x = k$
The way the curve approaches the asymptotes must also be detemined.

Examples

1. (If the degree of the numerator < the degree of the denominator) :-

e.g. $f(x) = \frac{2x+3}{x^2+5x+4} = \frac{2x+3}{(x+4)(x+1)}$

Vertical asymptotes occur at the zeros of the denominator

set $(x+4)(x+1) = 0$

=> $x = -4$ and $x = -1$ are asymptotes. (vertical lines)

We now have to determine how the curve approaches these asymptotes.
As $x \to -4^-$ (means x tends to -4 from a negative direction ie. $x < -4$)

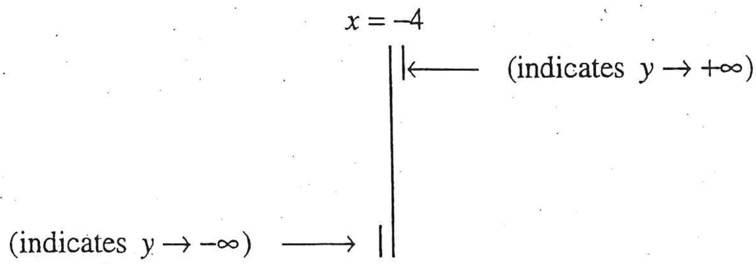
$y \to \frac{(-)}{(-)(-)} \to -\infty$ (assign a sign to each part of the rational function and find the net sign., $-\infty$ means negative infinity)

cont'd ....

As $x \to -4^+$ (means x tends to -4 from a positive direction ie. $x > -4$)

$y \to \dfrac{(-)}{(+)(-)} \to +\infty$ (assign a sign to each part of the rational function and find the <u>net sign</u>. $+\infty$ means positive infinity)

We can show this in an asymptote diagram.



As $x \to -1^-$ (means x tends to -1 from a negative direction ie. $x < -1$)

$y \to \dfrac{(+)}{(+)(-)} \to -\infty$ (assign a sign to each part of the rational function and find the <u>net sign</u>. $-\infty$ means negative infinity)

As $x \to -1^+$ (means x tends to -1 from a positive direction ie. $x > -1$)

$y \to \dfrac{(-)}{(+)(-)} \to +\infty$ (assign a sign to each part of the rational function and find the <u>net sign</u>. $+\infty$ means positive infinity)

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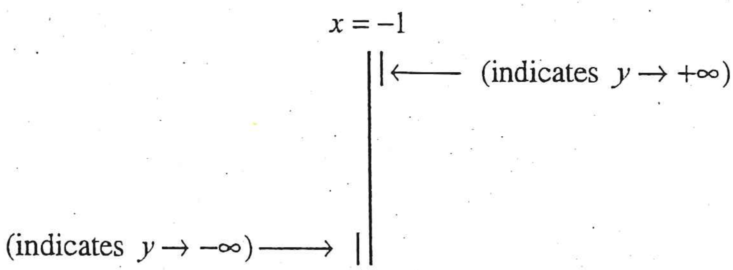


2. (If the degree of the numerator = the degree of the denominator) :-

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 5x + 4} = \frac{(x+1)^2}{(x+4)(x+1)}$$

<u>Vertical asymptotes occur at the zeros of the denominator</u>

$(x+4)(x+1) = 0$

$x = -4$ and $x = -1$ are asymptotes.

We now have to determine how the curve approaches these asymptotes.

As $x \to -4^-$ (means x tends to -4 from a negative direction ie. $x < -4$)

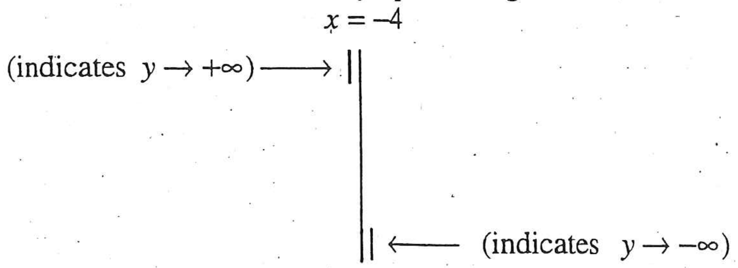
$y \to \dfrac{(+)}{(-)(-)} \to +\infty$ (assign a sign to each part of the rational function and find the <u>net sign</u>. $+\infty$ means positive infinity)

cont'd ....

As $x \to -4^{+}$ (means x tends to -4 from a positive direction ie. $x > -4$)

$y \to \dfrac{(+)}{(+)(-)} \to -\infty$ (put a sign to each part of the rational function and find the <u>net</u> <u>sign</u>. $-\infty$ means negative infinity)

We can show this in an asymptote diagram.



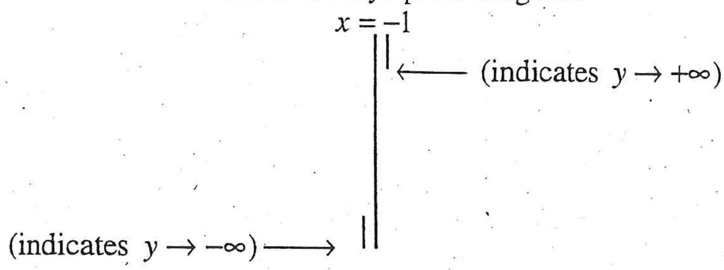
As $x \to -1^{-}$ (means x tends to -1 from a negative direction ie. $x < -1$)

$y \to \dfrac{(+)}{(+)(-)} \to -\infty$ (put a sign to each part of the rational function and find the <u>net</u> <u>sign</u>. $-\infty$ means negative infinity)

As $x \to -1^{+}$ (means x tends to -1 from a positive direction ie. $x > -1$)

$y \to \dfrac{(+)}{(+)(+)} \to +\infty$ (put a sign to each part of the rational function and find the <u>net</u> <u>sign</u>. $+\infty$ means positive infinity)

We can show this in an asymptote diagram.



3. (If the degree of the numerator > the degree of the denominator) :-

$$f(x) = \frac{x^2 + 4x + 3}{x + 2} = \frac{(x+1)(x+3)}{x+2}$$

<u>Vertical asymptotes occur at the zeros of the denominator</u>

$$x + 2 = 0$$

$x = -2$ is an asymptote.

We now have to determine how the curve approaches this asymptotes.

As $x \to -2^{-}$ (means x tends to -2 from a negative direction ie. $x < -2$)

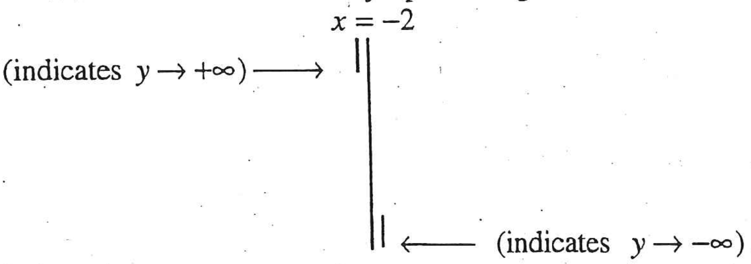
$y \to \dfrac{(-)(+)}{(-)} \to +\infty$ (put a sign to each part of the rational function and find the <u>net</u> <u>sign</u>. $+\infty$ means negative infinity)

cont'd ....

As $x \to -2^+$ (means x tends to -2 from a positive direction ie. $x > -2$)

$y \to \frac{(-)(+)}{(+)} \to -\infty$ (put a sign to each part of the rational function and find the <u>net sign</u>. $-\infty$ means positive infinity)

We can show this in an asymptote diagram.



<u>Exercise 1</u>

Find all the vertical asymptotes of the following rational functions.

1. $f(x) = \frac{4}{x-2}$
2. $f(x) = \frac{3x-1}{x^2+2x-3}$
3. $f(x) = \frac{12}{x^2-2x-3}$
4. $f(x) = \frac{x+4}{x-2}$
5. $f(x) = \frac{x^2}{4-x^2}$
6. $f(x) = \frac{x(x+1)}{(x-1)(x+2)}$
7. $f(x) = \frac{(x-1)(x-4)}{x}$
8. $f(x) = \frac{x^2+3}{x-1}$
9. $f(x) = \frac{x^3}{x^2+3}$
10. $f(x) = \frac{x}{x^2+4}$
11. $f(x) = \frac{x^2}{x-1}$
12. $f(x) = \frac{2x^2}{x^2-1}$

<u>Find the non-vertical asymptote of a rational function.</u>

The **non - vertical asymptote** is in the form $y = c$ or $y = mx + c$.
The way the curve approaches the asymptotes must also be determined.

<u>Examples</u>

1. (Note the degree of the numerator < the degree of the denominator)

$$f(x) = \frac{2x+3}{x^2+5x+4} = \frac{2x+3}{(x+4)(x+1)}$$

<u>Divide each term of the numerator and denominator by the highest power of</u> x.

ie. divide by x^2 $\qquad f(x) = \dfrac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{5}{x}+\frac{4}{x^2}}$

As $x \to \pm\infty$(ie large positive and negative values of x, $\frac{2}{x}$, $\frac{3}{x^2}$, $\frac{5}{x}$ and $\frac{4}{x^2}$, all tend to **zero**.)

As a result, $f(x) \to 0$ and so $y = 0$ is a horizontal asymptote. (the x - axis !)
Now we must find out if the curve approaches this asymptote from <u>above or below</u>.

cont'd ....

Proceed as follows.

As $x \to +\infty,\ y \to 0^+$ (above the value of 0, since the fraction will be very small and positive)

As $x \to -\infty,\ y \to 0^-$ (below the value of 0, since the fraction $\frac{2}{x}$ is negative and larger than the fraction $\frac{3}{x^2}$, making the numerator negative. At the same time, the denominator becomes 1 – fraction + fraction making the <u>net sign</u> positive.)

We can show this in an asymptote diagram.

Indicates $y \to 0^-$ (from below) → ═══════════ ← Indicates $y \to 0^+$ (from above)

2. (If the degree of the numerator = the degree of the denominator) :-

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 5x + 4} = \frac{(x+1)^2}{(x+4)(x+1)}$$

<u>Divide the numerator by the denominator</u> :-

Write as $f(x) = 1 - \frac{(3x+3)}{x^2 + 5x + 4}$ using long division

$$\begin{array}{r|l} & \quad 1 \\ x^2 + 5x + 4 & x^2 + 2x + 1 \\ & x^2 + 5x + 4 \\ \hline & -3x - 3 \end{array}$$

Now divide each term of $\frac{(3x+3)}{x^2 + 5x + 4}$ by the highest power of x

ie. divide by x^2

$$f(x) = 1 - \frac{\left(\frac{3}{x} + \frac{3}{x^2}\right)}{\left(1 + \frac{5}{x} + \frac{4}{x^2}\right)}$$

As $x \to \pm\infty$ (ie large positive and negative values of x, $\frac{3}{x}$, $\frac{3}{x^2}$, $\frac{5}{x}$ and $\frac{4}{x^2}$, all tend to **zero**.)

As a result, $f(x) \to 1$ and so $y = 1$ is a horizontal asymptote.

Now we must find out if the curve approaches this asymptote from above or below.

Proceed as follows.

As $x \to +\infty,\ y \to 1^-$ (below the value of 1, since the fraction will be very small and positive)

As $x \to -\infty,\ y \to 1^+$ (above the value of 1, since the fraction $\frac{3}{x}$ is negative and <u>larger</u> than the fraction $\frac{3}{x^2}$, making the numerator negative. At the same time, the denominator becomes approximately 1. $f(x) = 1 - ({}^{-}/_{1}) \Rightarrow 1^+ \Rightarrow$ slightly bigger than 1.

We can show this in an asymptote diagram.

Indicates $y \to 1^+$ (from above) → ═══════════ ← Indicates $y \to 1^-$ (from below)

3. (If the degree of the numerator > the degree of the denominator) :-

$$f(x)=\frac{x^2+4x+3}{x+2}=\frac{(x+1)(x+3)}{x+2}$$

Divide the numerator by the denominator :-

Write as $f(x)=x+2-\frac{1}{x+2}$ using long division

$$\begin{array}{r|l} & x+2 \\ \hline x+2 & x^2+4x+3 \\ & \underline{x^2+2x} \\ & \quad 2x+3 \\ & \quad \underline{2x+4} \\ & \quad\quad -1 \end{array}$$

Now divide each term of $\frac{1}{x+2}$ by the highest power of x.

ie. divide by x $\quad f(x)=x+2-\dfrac{\frac{1}{x}}{1+\frac{2}{x}}$

As $x\to\pm\infty$ (ie large positive and negative values of x, $\frac{1}{x}$ and $\frac{2}{x}$ tend to **zero**.)

As a result, $f(x)\to x+2$ and so $f(x)=x+2$ is a oblique asymptote. (slanting)

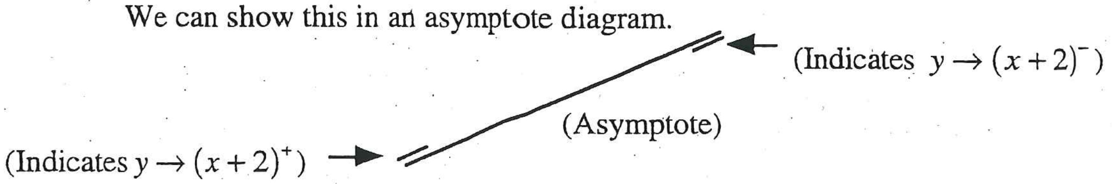
Now we must find out if the curve approaches this asymptote from above or below.

Proceed as follows.

As $x\to+\infty$, $y\to(x+2)^-$ (below the line $f(x)=x+2$, since the fraction will be very small and positive)

As $x\to-\infty$, $y\to(x+2)^+$ (above the line $f(x)=x+2$, since the fraction $\frac{1}{x}$ is negative and the denominator becomes $1-$ fraction making the net sign positive.)

We can show this in an asymptote diagram.



Exercise 2

Find the non - vertical asymptote of the following rational functions.

1. $f(x)=\frac{4}{x-2}$
2. $f(x)=\frac{3x-1}{x^2+2x-3}$
3. $f(x)=\frac{12}{x^2-2x-3}$
4. $f(x)=\frac{x+4}{x-2}$
5. $f(x)=\frac{x^2}{4-x^2}$
6. $f(x)=\frac{x(x+1)}{(x-1)(x+2)}$
7. $f(x)=\frac{(x-1)(x-4)}{x-2}$
8. $f(x)=\frac{x^2+3}{x-1}$
9. $f(x)=\frac{x^3}{x^2+3}$
10. $f(x)=\frac{x}{x^2+4}$
11. $f(x)=\frac{x^2}{x-1}$
12. $f(x)=\frac{2x^2}{x^2-1}$

Sketch the graph of a rational function including appropriate analysis of stationary points.

Curve Sketching

1 Sketching the graph of a rational function using calculus

(a) Find all the stationary points and their nature.

(b) Find all the asymptotes and investigate the approach of the curve to each.

(c) Find all the crossings of the y axis. and the x axis (if easily found)

Note The table of signs may be easier than finding the second derivative in order to determine the nature of the stationary points.

Examples Sketch the following curves.

1. $f(x)=\dfrac{2x^2+x-1}{x-1}=\dfrac{(x+1)(2x-1)}{x-1}$

(a) Stationary points and their nature.

$$f'(x)=\frac{(x-1)(4x+1)-\left(2x^2+x-1\right)\times 1}{(x-1)^2}=\frac{4x^2-3x-1-2x^2-x+1}{(x-1)^2}=\frac{2x^2-4x}{(x-1)^2}$$

For S.V. $f'(x)=0$

$2x^2-4x=0$

$2x(x-2)=0$

$x=0$ and $x=2$

$y=1$ $\quad y=9$

Stationary points at (0,1) and (2,9)

$$f''(x)=\frac{(x-1)^2\times(4x-4)-\left(2x^2-4x\right)\times 2(x-1)}{(x-1)^4}=\frac{4(x-1)^3-4x(x-1)(x-2)}{(x-1)^4}$$

$$=\frac{4(x-1)^2-4x(x-2)}{(x-1)^3}$$

$f''(0)=\dfrac{(+)-0}{(-)}=(-)$ ie. (0, 1) is a Maximum Turning point.

$f''(2)=\dfrac{(+)-0}{(+)}=(+)$ ie. (2, 9) is a Minimum Turning point.

(b) Asymptotes.

(i) Vertical.

Vertical asymptotes occur at the zeros of the denominator

$x-1=0$

$x=1$ is an asymptote.

$x=1$

As $x\to1^-$, $y\to\dfrac{(+)(+)}{(-)}\to-\infty$

As $x\to1^+$, $y\to\dfrac{(+)(+)}{(+)}\to+\infty$

cont'd ....

(ii) **Non-vertical.**

Divide the numerator by the denominator :-

Write as $f(x) = 2x + 3 + \dfrac{2}{x-1}$ using long division

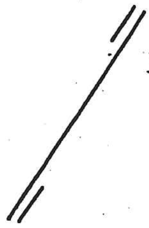
$$\begin{array}{r|l} & \quad\;\; 2x+3 \\ x-1 & 2x^2+x-1 \\ & 2x^2-2x \\ & \qquad\; 3x-1 \\ & \qquad\; 3x-3 \\ & \qquad\quad\;\; +2 \end{array}$$

$$f(x) = 2x + 3 + \frac{\frac{2}{x}}{1-\frac{1}{x}}$$

As $x \to \pm\infty$, $f(x) \to 2x+3$ and so $f(x) = 2x+3$ is a slant asymptote.

As $x \to +\infty$, $y \to (2x+3)^+$ (above)

As $x \to -\infty$, $y \to (2x+3)^-$ (below).



$y = 2x + 3$

(c) **Axes crossings.**

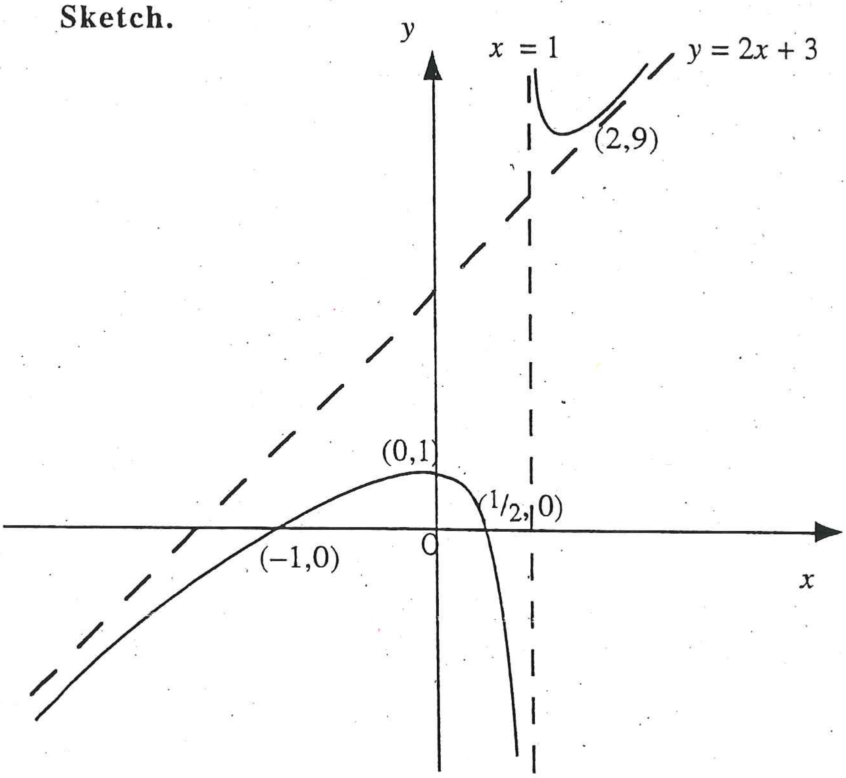
When $f(x) = 0$, $(x+1)(2x-1) = 0$ ie. $x = -1$ and $x = \dfrac{1}{2}$.

The graph crosses the x-axis at $(-1,0)$ and $\left(\dfrac{1}{2}, 0\right)$

When $x = 0$, $y = 1$.

The graph crosses the y-axis at $(0, 1)$.

Sketch.



2. $f(x)=\dfrac{2x^2+4x+3}{x^2-1}=\dfrac{2x^2+4x+3}{(x+1)(x-1)}$

(a) **Stationary points and their nature.**

$$f'(x)=\frac{(x^2-1)(4x+4)-(2x^2+4x+3)\times 2x}{(x^2-1)^2}=\frac{4x^3+4x^2-4x-4-4x^3-8x^2-6x}{(x^2-1)^2}$$
$$=\frac{-4x^2-10x-4}{(x^2-1)^2}$$

For S.V. $f'(x)=0$

$-4x^2-10x-4=0$

$-2(x+2)(2x+1)=0$

$x=-2$ and $x=-\dfrac{1}{2}$

$y=1$ $\quad$ $y=-2$

Stationary points at $(-2,1)$ and $(-\frac{1}{2},-2)$

$$f''(x)=\frac{(x^2-1)^2(-8x-10)-(-4x^2-10x-4)4x(x^2-1)}{(x^2-1)^4}$$
$$=\frac{-2(x^2-1)(4x+5)+8x(x+2)(2x+1)}{(x^2-1)^3}$$

$f''(-2)=\dfrac{(+)+0}{(+)}=(+)$ ie. $(-2,1)$ is a Minimum Turning point.

$f''\left(-\dfrac{1}{2}\right)=\dfrac{(+)+0}{(-)}=(-)$ ie. $\left(-\dfrac{1}{2},-2\right)$ is a Maximum Turning point.

(b) **Asymptotes.**

(i) **Vertical.**

Vertical asymptotes occur at the zeros of the denominator

$(x+1)(x-1)=0$

$x=-1$ and $x=1$ are asymptotes.

$x=-1$

As $x\to -1^-$, $y\to\dfrac{(+)}{(-)(-)}\to +\infty$

As $x\to -1^+$, $y\to\dfrac{(+)}{(+)(-)}\to -\infty$

$x=1$

As $x\to 1^-$, $y\to\dfrac{(+)}{(+)(-)}\to -\infty$

As $x\to 1^+$, $y\to\dfrac{(+)}{(+)(+)}\to +\infty$

cont'd ....

(ii) Non-vertical.

Divide the numerator by the denominator :-

Write as $f(x) = 2 + \dfrac{4x+5}{x^2-1}$ using long division

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2-1 & 2x^2+4x+3 \\ & 2x^2 \quad\quad -2 \\ \hline & \quad\quad 4x+5 \end{array}$$

$$f(x) = 2 + \frac{\frac{4}{x} + \frac{5}{x^2}}{1 - \frac{1}{x^2}}$$

As $x \to \pm\infty$, $f(x) \to 2$ and so $f(x) = 2$ is a horizontal asymptote.

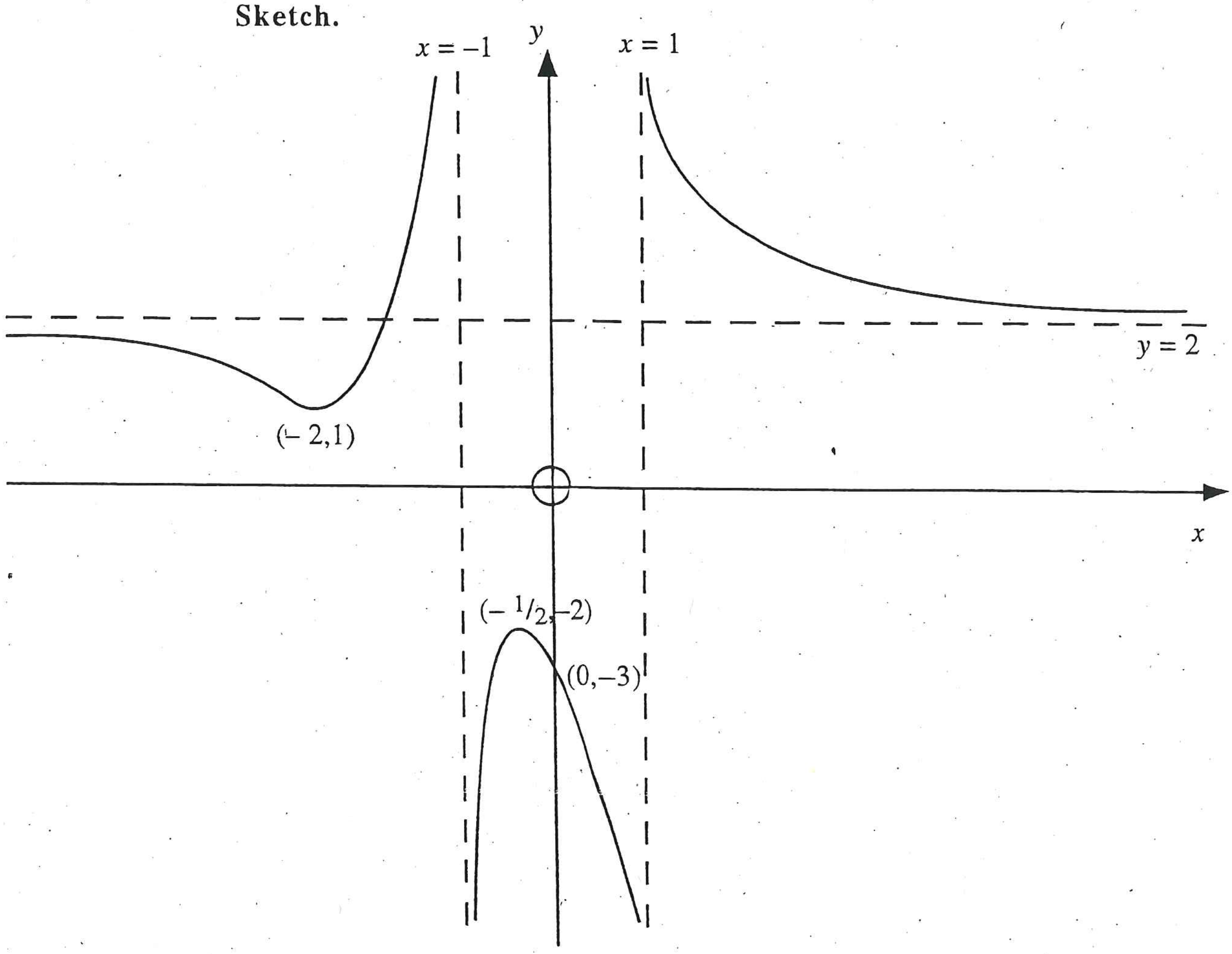
As $x \to +\infty$, $y \to 2^+$ (above)

As $x \to -\infty$, $y \to 2^-$ (below)

(c) Axes crossings.

When $f(x)=0$, $2x^2 + 4x + 3 = 0$ ie. has no real roots since $b^2 - 4ac < 0$.
The graph does not cross the x-axis .
When $x = 0$, $y = -3$.
The graph crosses the y-axis at $(0,-3)$.

Sketch.



Exercise 3

Sketch the following curves.

1. $f(x)=\dfrac{4}{x-2}$

2. $f(x)=\dfrac{x-2}{x-1}$

3. $f(x)=\dfrac{x^2+x-2}{x^2+x-6}$

4. $f(x)=\dfrac{x^2+2x+5}{x+1}$

5. $f(x)=\dfrac{x+1}{x^2+2x+5}$

6. $f(x)=\dfrac{2x^2}{x^2-1}$

7. $f(x)=\dfrac{x^2-10x+9}{x^2+10x+9}$

8. $f(x)=\dfrac{2x^2-3x-3}{x^2-3x+2}$

2 Sketching the graph of a rational function without using calculus

(a) Find the crossings of the axes.

(b) Find the equations of the asymptotes and investigate how the curve approaches each.

(c) Find the **range** of values of y.

Examples

1. Sketch the curve $y=\dfrac{3x+3}{x(3-x)}$

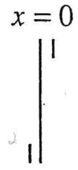
(a) At $y=0$, $\dfrac{3x+3}{x(3-x)}=0$ => $3x+3=0$ => $x=-1$ ie. $(-1,0)$

At $x=0$, $y=\dfrac{3}{0}$ (undefined) ie. does not cross the y - axis

(b) **Vertical asymptotes** at $x(3-x)=0$ ie. $x=0$, $x=3$

As $x\to 0^-, y\to \dfrac{(+)}{(-)(+)}\to -\infty$

As $x\to 0^+, y\to \dfrac{(+)}{(+)(+)}\to +\infty$



As $x\to 3^-, y\to \dfrac{(+)}{(+)(+)}\to +\infty$

As $x\to 3^+, y\to \dfrac{(+)}{(+)(-)}\to -\infty$

Horizontal/sloping asymptotes

$$y=\frac{3x+3}{3x-x^2}=\frac{\frac{3x}{x^2}+\frac{3}{x^2}}{\frac{3x}{x^2}-\frac{x^2}{x^2}}=\frac{\frac{3}{x}+\frac{3}{x^2}}{\frac{3}{x}-1}$$

As $x\to\pm\infty$, $y\to 0$ and so $y=0$ is a horizontal asymptote.

As $x\to +\infty, y\to \dfrac{(+)}{-1}\to 0^-$

As $x\to -\infty, y\to \dfrac{(-)}{-1}\to 0^+$

cont'd ....

(c) **Range of the values of y**

From $y = \dfrac{3x+3=0}{x(3-x)}$

$3x+3 = y(3x - x^2) \Rightarrow 3x+3 = 3xy - x^2y$

$x^2y + 3x - 3xy + 3 = 0 \Rightarrow x^2y + 3x(1-y) + 3 = 0$

$x^2y + 3x(1-y) + 3 = 0$ is a quadratic equation in x.

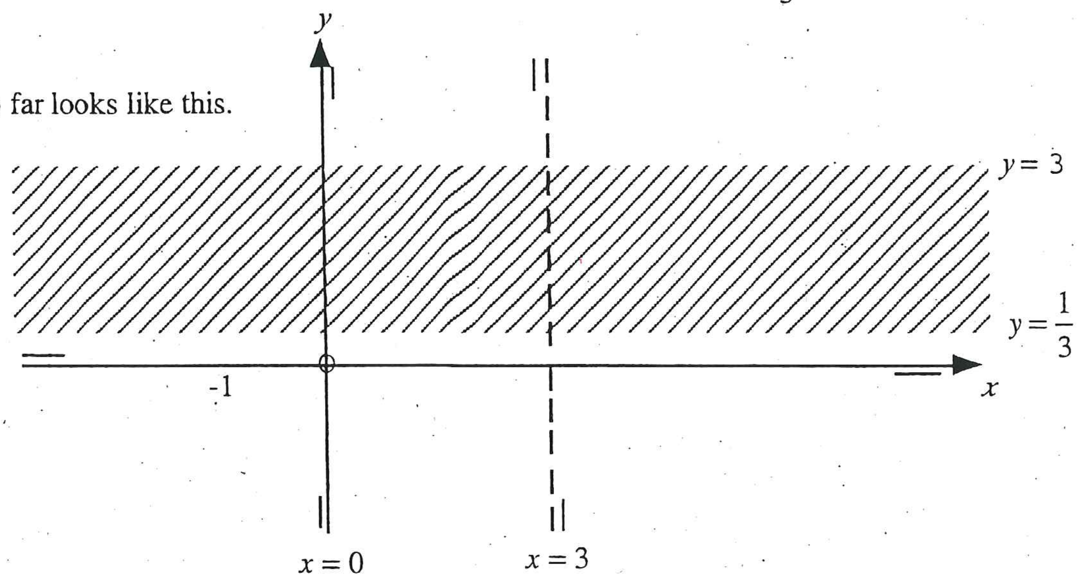
Thus for real values of x, $b^2 - 4ac \geq 0$.

ie. $[3(1-y)]^2 - 12y \geq 0$ or $9y^2 - 30y + 9 \geq 0$ or $3(3y-1)(y-3) \geq 0$

ie. $y \leq \frac{1}{3}$ and $y \geq 3$ for real x.

This means that no part of the graph can lie between $y = \frac{1}{3}$ and $y = 3$.

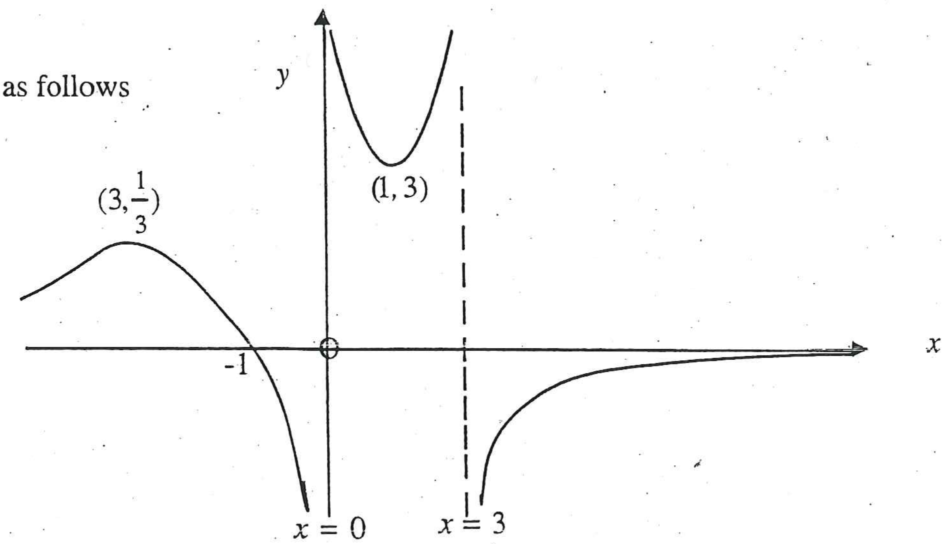
The curve so far looks like this.



We expect a (local) maximum for $y \leq \frac{1}{3}$ and a (local) minimum for $y \geq 3$.

The coordinates of the turning points can be found by substitution of $y = \frac{1}{3}$ and $y = 3$ into $x^2y + 3x(1-y) + 3 = 0$ to find the corresponding values of x ie. -3 and 1.

The complete graph is as follows



2. Sketch $y = \dfrac{12}{x^2 + 2x - 3} = \dfrac{12}{(x+3)(x-1)}$

(a) When $y = 0$, $\dfrac{12}{(x+3)(x-1)} = 0$ (No solution ie. does not cross the x-axis)

When $x = 0$, $y = -4$ Cosses the y-axis at $(0, -4)$

(b) **Vertical asymptotes** at $(x+3)(x-1) = 0$ ie. $x = -3$ and $x = 1$

$x = -3$

As $x \to -3^-, y \to \dfrac{(+)}{(-)(-)} \to +\infty$

As $x \to -3^+, y \to \dfrac{(+)}{(+)(-)} \to -\infty$

$x = 1$

As $x \to 1^-, y \to \dfrac{(+)}{(+)(-)} \to -\infty$

As $x \to 1^+, y \to \dfrac{(+)}{(+)(+)} \to +\infty$

Horizontal/slant asymptotes

$$y = \frac{12}{x^2 + 2x - 3} = \frac{\frac{12}{x^2}}{\frac{x^2}{x^2} + \frac{2x}{x^2} - \frac{3}{x^2}} = \frac{\frac{12}{x^2}}{1 + \frac{2x}{x^2} - \frac{3}{x^2}}$$

As $x \to \pm\infty$, $y \to 0$ and so $y = 0$ is a horizontal asymptote.

$y = 0$

As $x \to +\infty, y \to \dfrac{(+)}{1} \to 0^+$

As $x \to -\infty, y \to \dfrac{(+)}{1} \to 0^+$

(c) **Range of the values of y**

From $y = \dfrac{12}{x^2 + 2x - 3}$

$y(x^2 + 2x - 3) = 12 \Rightarrow x^2y + 2xy - 3y - 12$

$x^2y + 2xy - (3y + 12) = 0$

$x^2y + 2xy - (3y + 12) = 0$is a quadratic equation in x.

Thus for real values of x , $b^2 - 4ac \geq 0$.

ie. $4y^2 + 4y(3y + 12) \geq 0$

$16y^2 + 48y \geq 0$

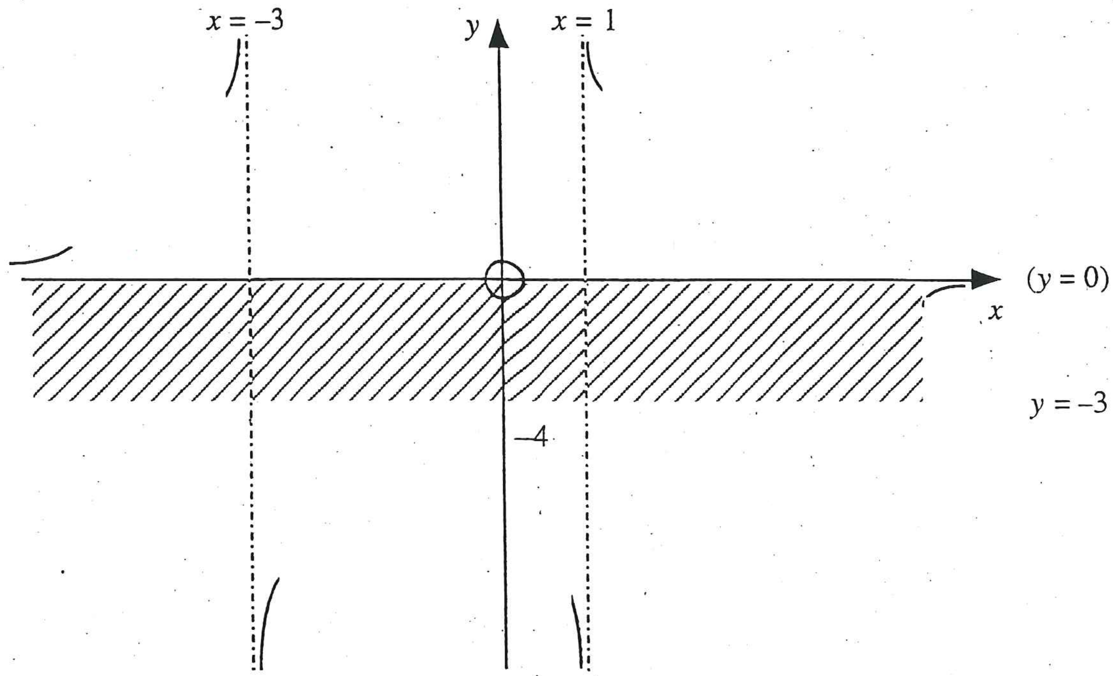
$16y(y + 3) \geq 0$

Solve by either a table of signs or sketching the quadratic graph.

ie. $y \leq -3$ and $y \geq 0$

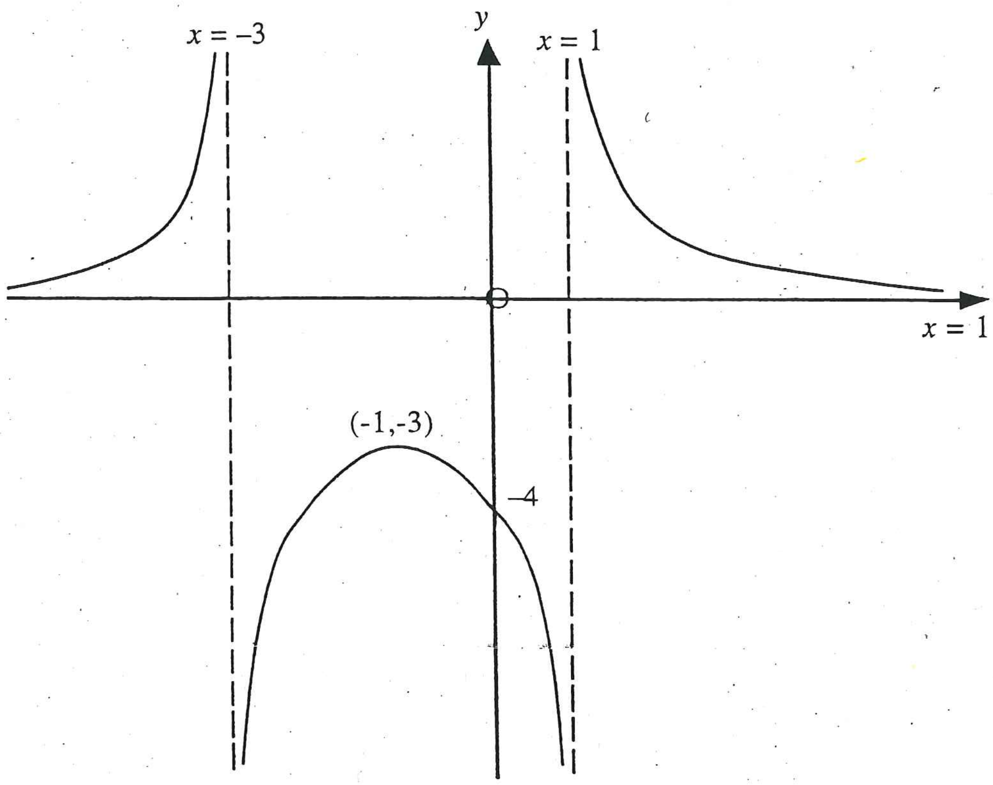
This means that no part of the graph can lie between $y = -3$ and $y = 0$
The curve so far looks like this.

cont'd ....



For $y \leq -3$, there is a (local) maximum.
The coordinates of the turning point can be found by substituting $y = -3$ into $x^2y + 2xy - (3y + 12) = 0$ to find the corresponding value of x i.e. $x = -1$

The completed curve looks as follows.



Exercise 4

Sketch the following curves.

1. $y = \dfrac{4x-5}{x^2-1}$

2. $y = \dfrac{x^2+3}{x-1}$

3. $y = \dfrac{4(x+1)}{x^2+2x+2}$

4. $y = \dfrac{2x-1}{(x-1)^2}$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 93 Exercise 2.3:1 Questions 1, 2,7,18,19 and 22

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 342 Exercise 14A Questions 9-19.

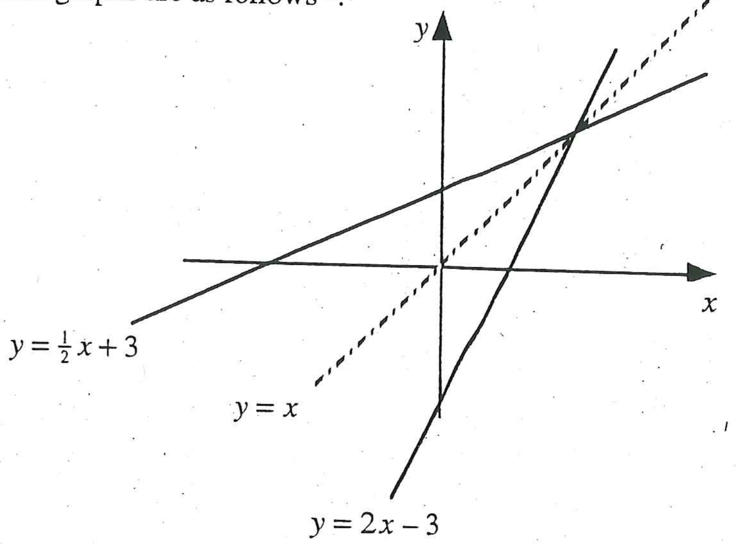
Given the graph of a function, sketch the graphs of a related function.

(a) The graph of inverse functions

The graph of the inverse of simple functions has already been met at Higher level.

e.g. (i) $f(x) = 2x - 3$ has an inverse of $f^{-1}(x) = \frac{1}{2}(x+3)$ or $\frac{1}{2}x + \frac{3}{2}$

The graphs are as follows -:

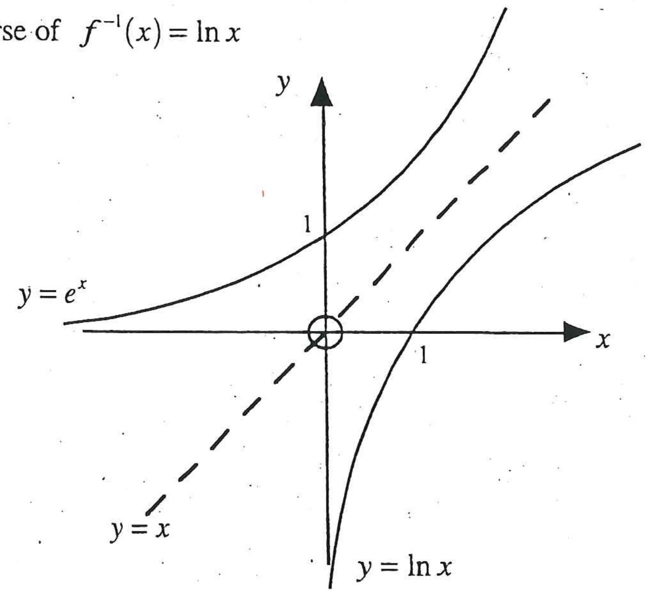


It has already been shown that the graph of an inverse function can be found by reflecting the graph of the function in the line $y = x$.

cont'd ....

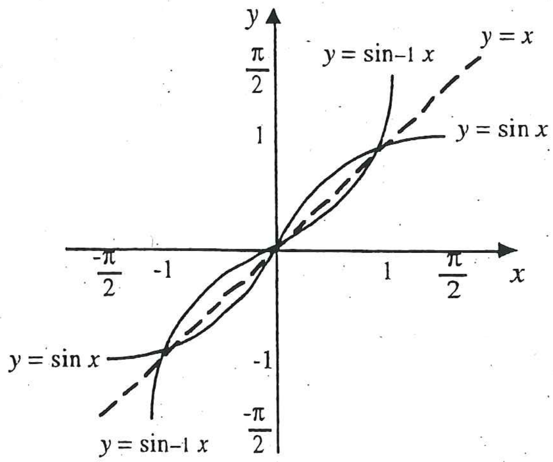
(ii) $f(x) = e^x$ has an inverse of $f^{-1}(x) = \ln x$

The graphs are as follows -:



(iii) $f(x) = \sin x$ has an inverse of $f^{-1}(x) = \sin^{-1} x$

The graphs are as follows -:



The inverse Sine Function is denoted by $\sin^{-1} x$ and read as sine inverse x or arc sine x or sine to the minus 1 of x.

The inverse sine function , $\sin^{-1} x$, is defined as the **angle** whose sine is x.

If we consider the graph of $y = \sin x$, we see that there is an infinite number of angles whose sine could be x.

Consequently , in order that the inverse sine should be a true function , we must restrict the angle concerned.

We choose the simplest possible restriction for the angle - the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

A revised definition is therefore that the inverse sine function , $\sin^{-1} x$,is the angle, in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, whose sine is x.

Examples

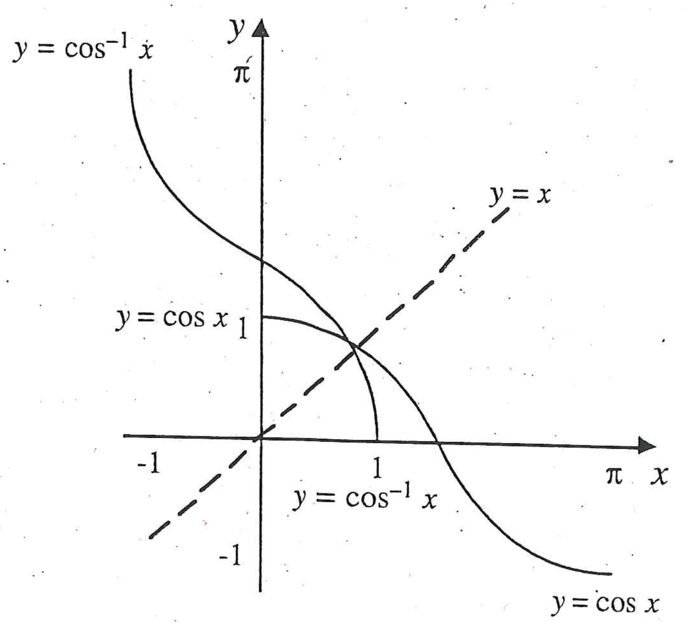
Evaluate :-

1. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$ 2. $\sin^{-1}(-1)=-\frac{\pi}{2}$

3. $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$ 4. $\sin^{-1}(0)=0$

(iv) $f(x)=\cos x$ has an inverse of $f^{-1}(x)=\cos^{-1}x$

The graphs are as follows -:



The inverse Cosine Function is denoted by $\cos^{-1}x$ and is read as cos inverse x or arc cos x or cos to the minus 1 of x.

In this case , we restrict the angle to the closed interval $[0,\pi]$.

The inverse cosine function is defined as the angle , in the closed interval $[0,\pi]$, whose cosine is x.

Examples

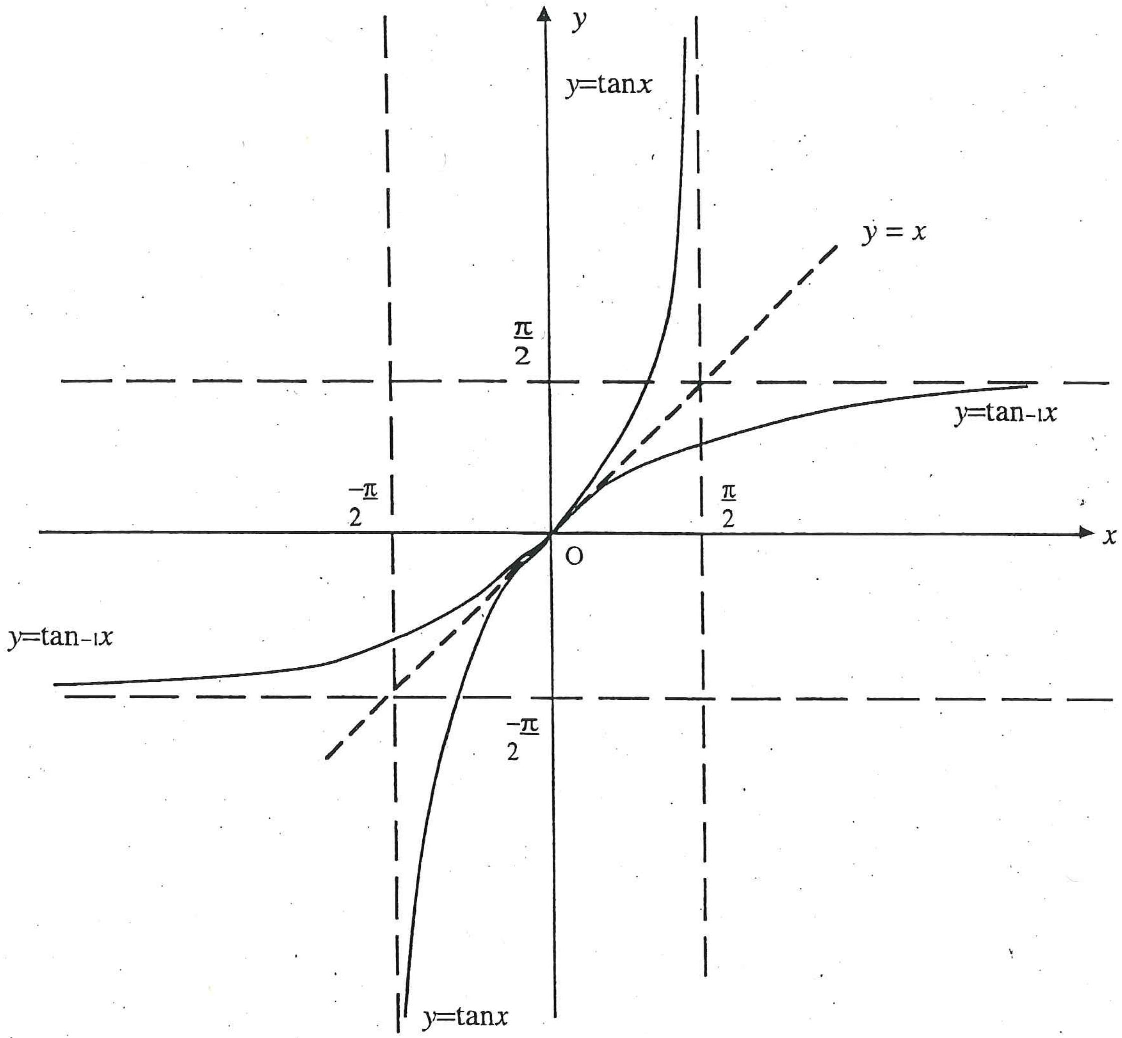
Evaluate :-

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$ 2. $\cos^{-1}(-1)=\pi$

cont'd ....

(v) $f(x) = \tan x$ has an inverse of $f^{-1}(x) = \tan^{-1} x$.

The graphs are as follows -:



The inverse Tangent function is denoted by $\tan^{-1} x$ and is read as tan inverse x or arc tan x. In this case we restrict the angle to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The inverse tangent function is defined as the angle in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

Examples

Evaluate :-

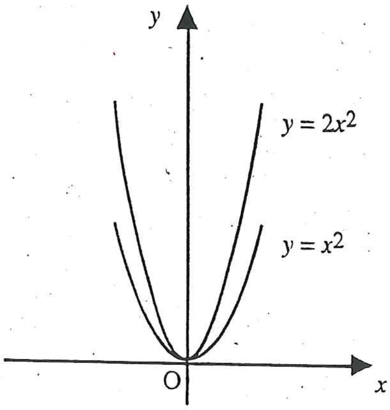
1. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$

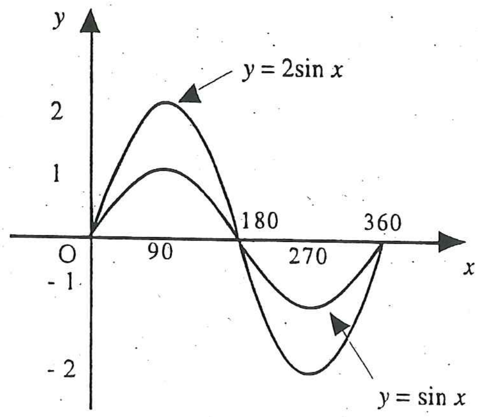
2. $\tan^{-1}(-1) = -\frac{\pi}{4}$

The following relationships on the next page were met at Higher level

cont'd ....

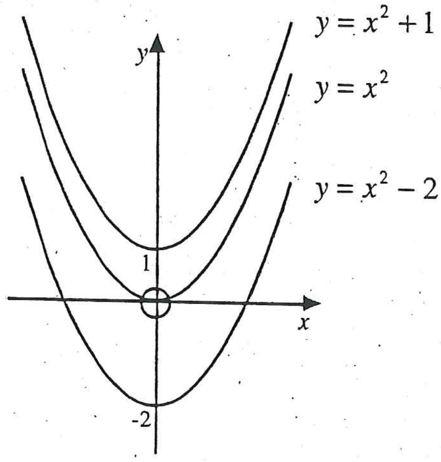
(b) The graph of $kf(x)$

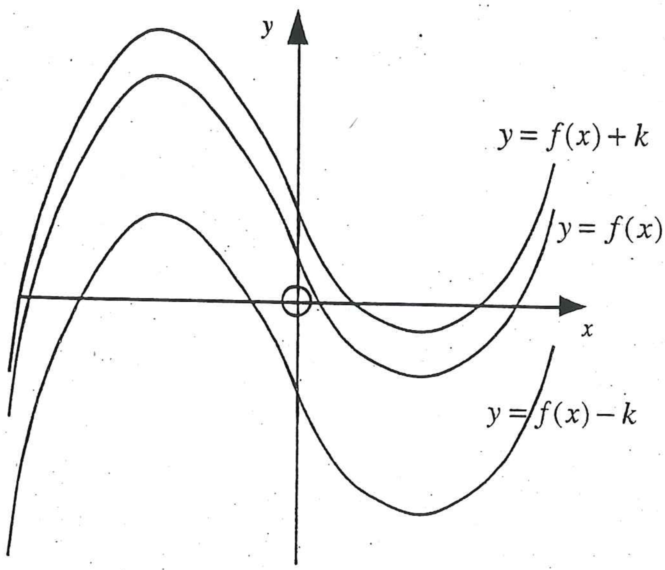




The graph of $f(x)$ is stretched vertically if $k > 1$ and compressed vertically if $0 < k < 1$.

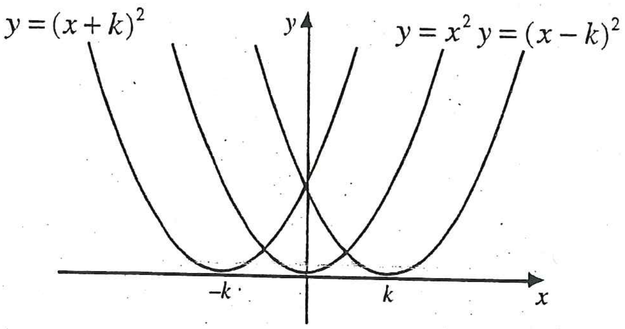
(c) The graph of $f(x) + k$

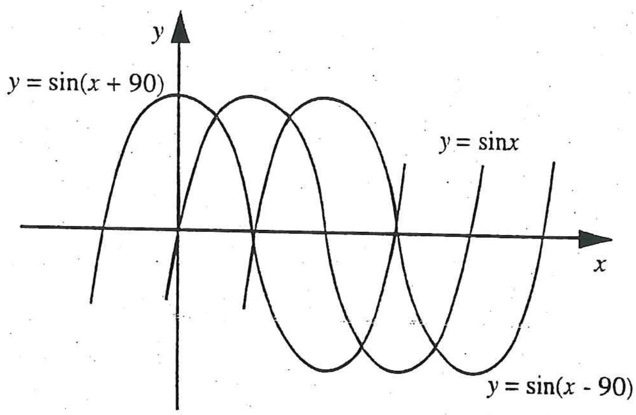




The graph of $f(x)$ is moved vertically upwards if $k > 0$ and downwards if $k < 0$.

(d) The graph of $f(x + k)$

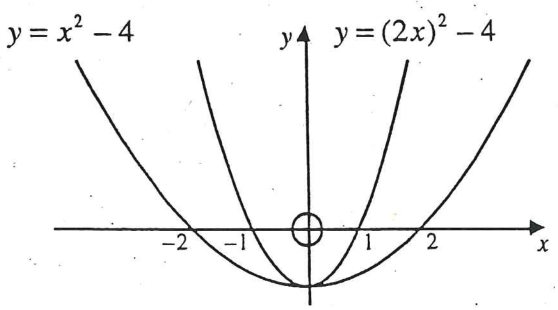


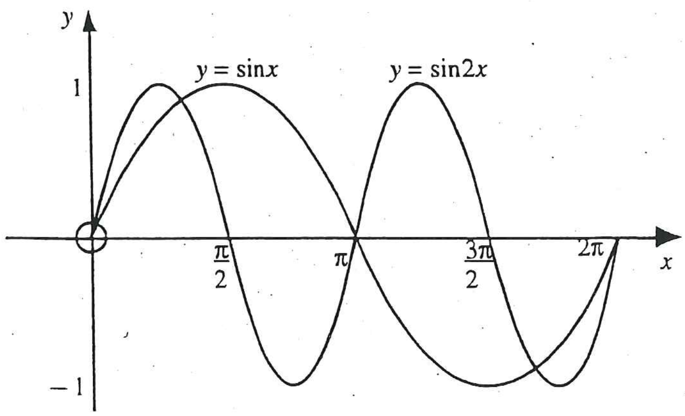


The graph of $f(x)$ is moved horizontally left if $k > 0$ and right if $k < 0$.

cont'd ....

(e) The graph of $f(kx)$

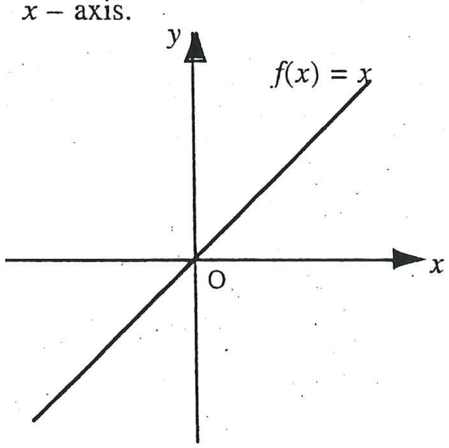


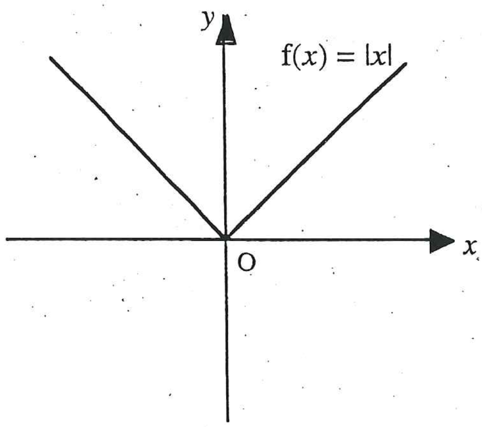


The graph of $f(x)$ is compressed if $k > 1$ and stretched if $k < 1$.

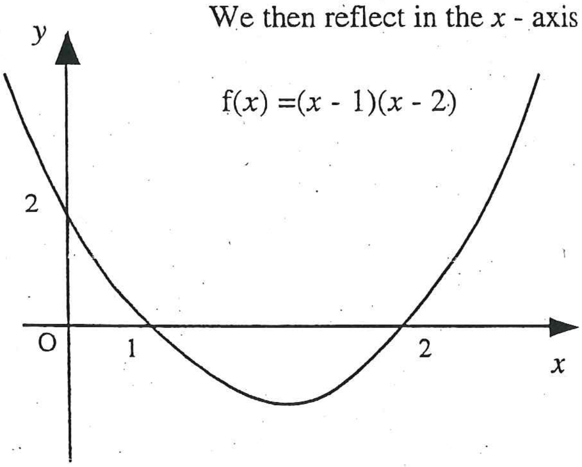
(f) The graph of $|f(x)|$ - The Modulus Function

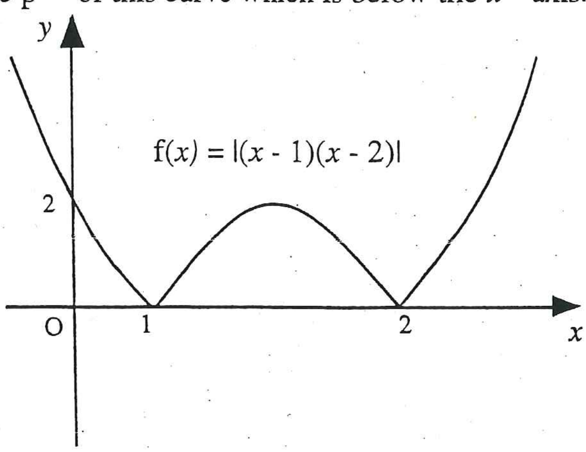
(i) When $f(x) = x$, $f(x)$ is negative when x is negative.
But if $f(x) = |x|$, f takes the **positive** numerical value of x.
e.g. when $x = -3$, $|x| = 3$ so $f(x)$ is always positive.
The graph of $f(x) = |x|$ can be obtained from the graph of $f(x) = x$ by simply flipping any parts of the graph of $f(x)$, which appear below the x – axis, over the x – axis.





(ii) To sketch $f(x) = |(x-1)(x-2)|$, we start by sketching $f(x) = (x-1)(x-2)$.
We then reflect in the x - axis the p of this curve which is below the x – axis.

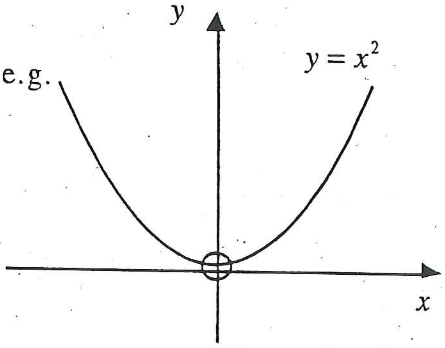


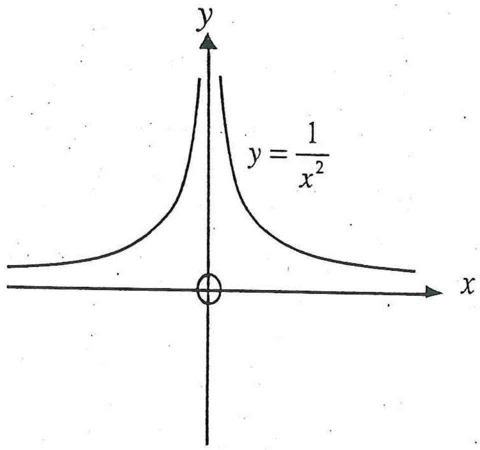


(g) Odd and Even functions

An **even** function is any function whose curve has the y - axis as a line of symmetry. Curves, having only even powers of x, are symmetrical about the y - axis.

e.g.





An alternative method of defining an **even** function is to show that $f(-x) = f(x)$ for **all** values of x.

e.g. (i) $f(x) = x^2$

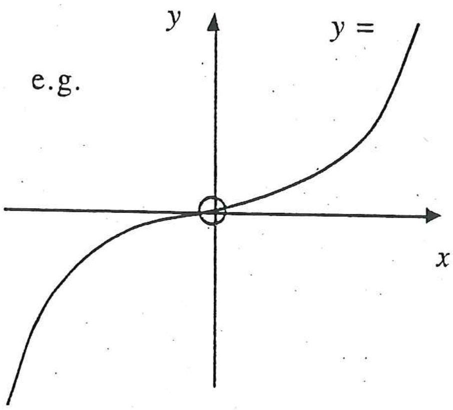
$f(-x) = (-x)^2 = f(x)$

(ii) $f(x) = \dfrac{1}{x^2}$

$f(-x) = \dfrac{1}{(-x)^2} = \dfrac{1}{x^2} = f(x)$

An **odd** function is any function whose curve has 180° rotational symmetry about the origin. Curves having only odd powers of x have 180° rotational symmetrical about the origin.

e.g.



An alternative method of defining an **odd** function is to show that $f(-x) = -f(x)$. for **all** values of x.

e.g. $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

Exercise 5

1. Write down the equations of the inverses of the following functions :

   (a) $f(x) = 2x$ (b) $f(x) = 2 - x$ (c) $f(x) = {}^2/x$

   (d) $f(x) = 2^x$ (e) $f(x) = 1 - 2x$ (f) $f(x) = \ln(x - 2)$

2. Evaluate :

   (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\tan^{-1}(1/\sqrt{3})$ (c) $\tan^{-1}(1)$

   (d) $\sin^{-1}(1/2)$ (e) $\cos^{-1}(-\sqrt{3}/2)$ (f) $\tan^{-1}(\sqrt{3})$

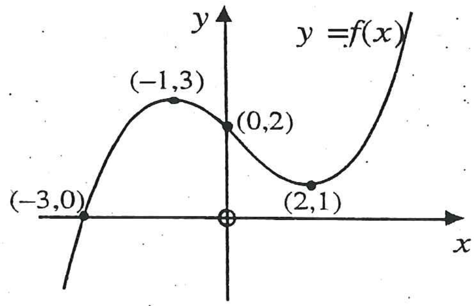
3. Sketch the following graph of a function $y = f(x)$ for each part of the following question.

   On separate graphs, sketch the graphs of :

   (a) $f(x - 3)$ (b) $f(x + 3)$

   (c) $f(x) + 2$ (d) $2f(x)$

   (e) $-f(x)$ (f) $f(-x)$



4. Sketch the graphs of $f(x)$ and $|f(x)|$.

   (a) $f(x) = x + 2$ (b) $f(x) = 5 - 2x$ (c) $f(x) = x^2 - 2x - 3$

   (d) $f(x) = 3x - x^2$ (e) $f(x) = x^3 + 1$ (f) $f(x) = \dfrac{1}{x} - 2$

5. Which of the following functions are odd, even or neither ?

   (a) $f(x) = (x + 4)(x - 2)$ (b) $f(x) = 3x^2 + 5$

   (c) $f(x) = 2x - x^3$ (d) $f(x) = \sin 2x$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Various sections on graphs.Page 105, 108, 128, 163.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Various sections on graphs.Page 45, 287, 290, 342, 348.

Answers

Exercise 1 Page 50

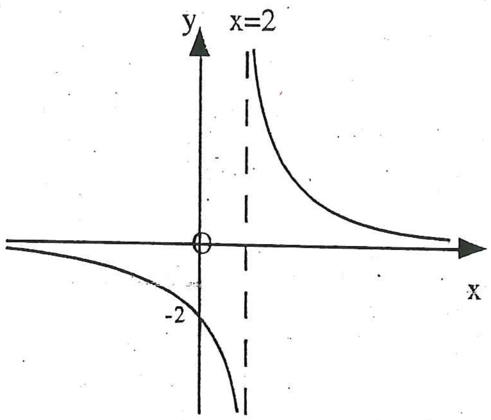
1. $x = 2$
2. $x = -3$ $x = 1$
3. $x = -1$ $x = 3$
4. $x = 2$
5. $x = -2$ $x = 2$
6. $x = -2$ $x = 1$
7. $x = 0$
8. $x = 1$
9. None
10. None
11. $x = 1$
12. $x = -1$ $x = 1$

Exercise 2 Page 52

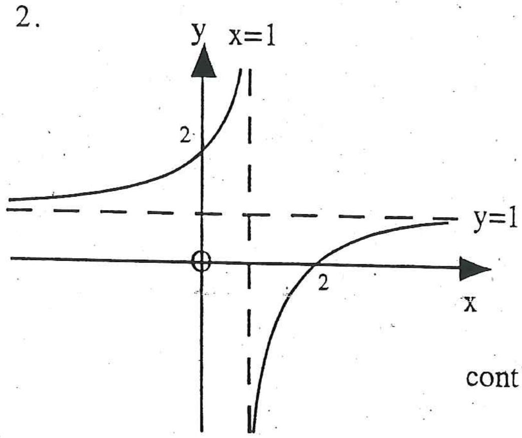
1. $y = 0$
2. $y = 0$
3. $y = 0$
4. $y = 1$
5. $y = -1$
6. $y = 1$
7. $y = x - 3$
8. $y = x + 1$
9. $y = x$
10. $y = 0$
11. $y = x + 1$
12. $y = 2$

Exercise 3 Page 57

1.



2.



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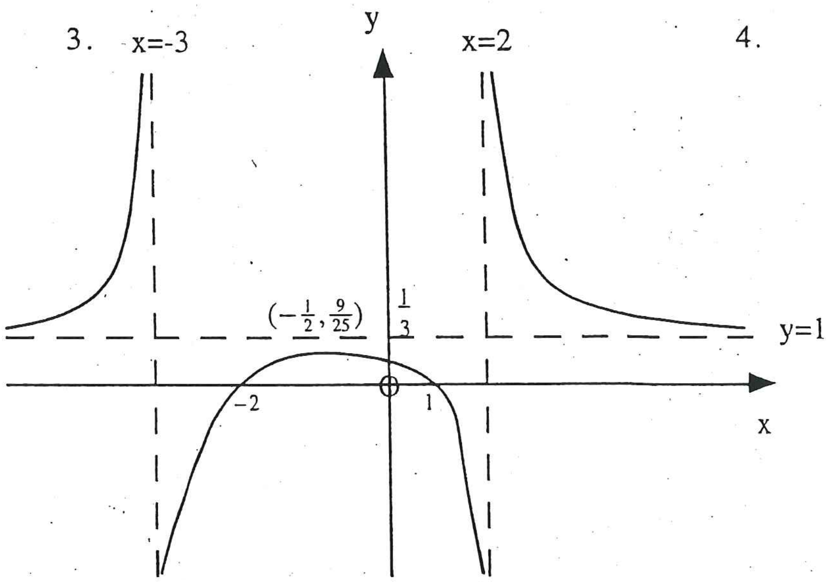
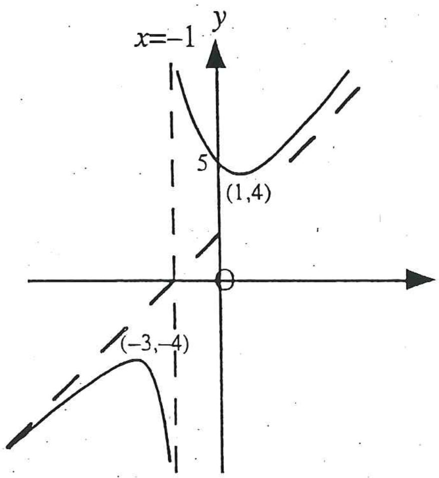
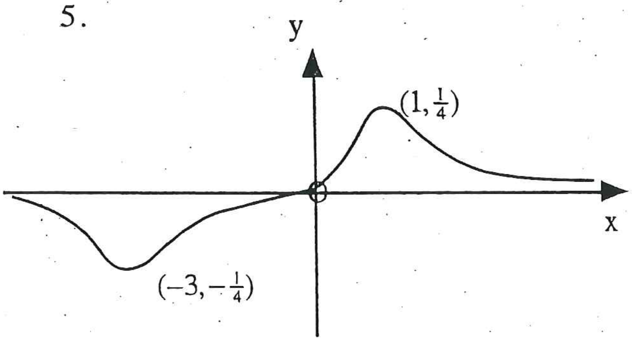
3.


x=-3
y
x=2
$(-\frac{1}{2}, \frac{9}{25})$
$\frac{1}{3}$
y=1
-2
1
x

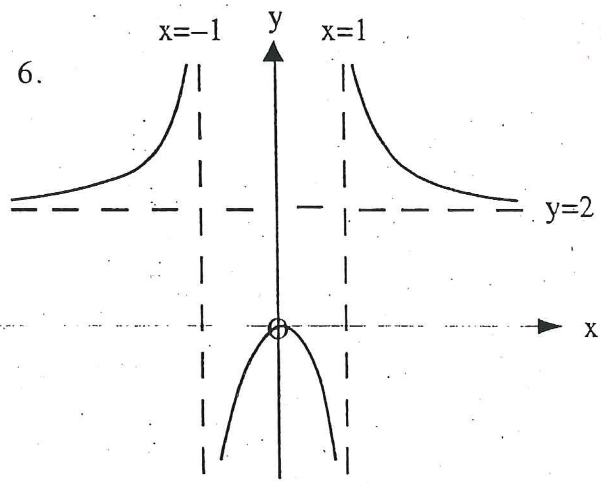
4.


x=-1
y
5
(1,4)
(-3,-4)

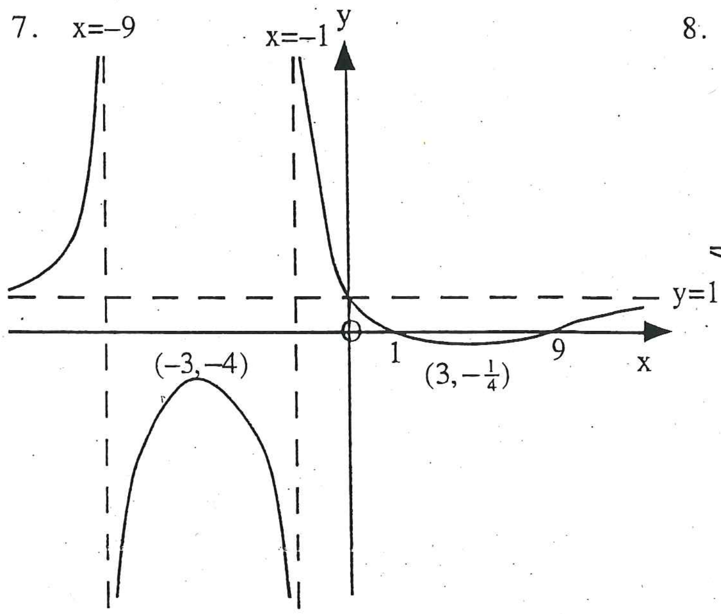
5.


y
$(1,\frac{1}{4})$
x
$(-3,-\frac{1}{4})$

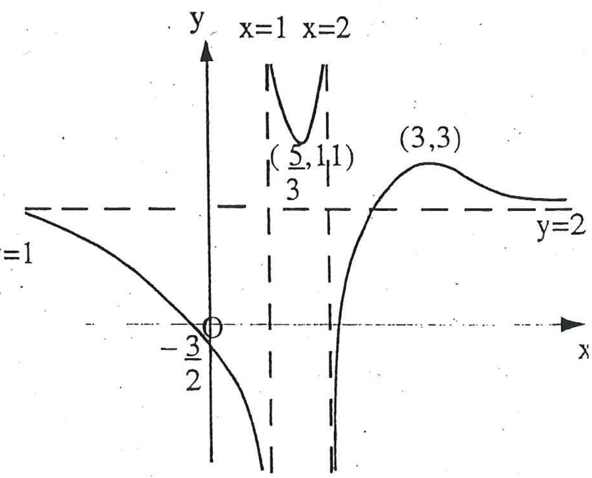
6.


x=-1
y
x=1
y=2
x

7.

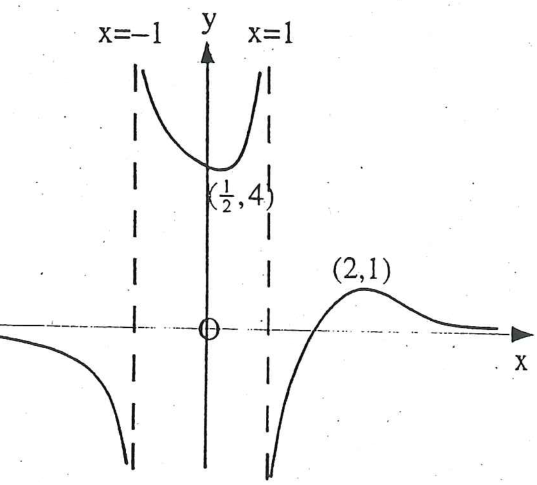

x=-9
x=-1
y
y=1
(-3,-4)
1
$(3,-\frac{1}{4})$
9
x

8.

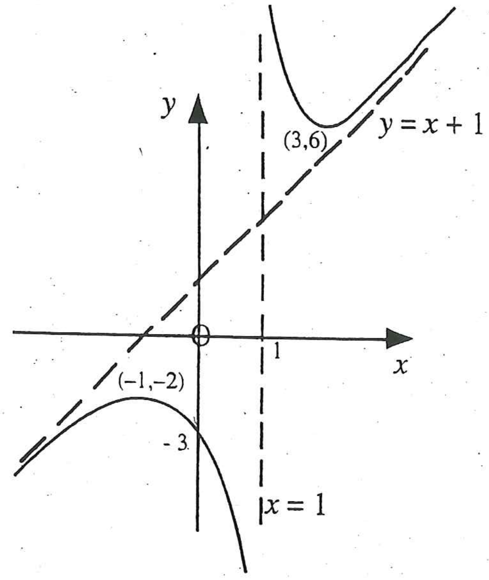

y
x=1 x=2
$(\frac{5}{3},11)$
(3,3)
y=2
$-\frac{3}{2}$
x

Exercise 4 Page 61

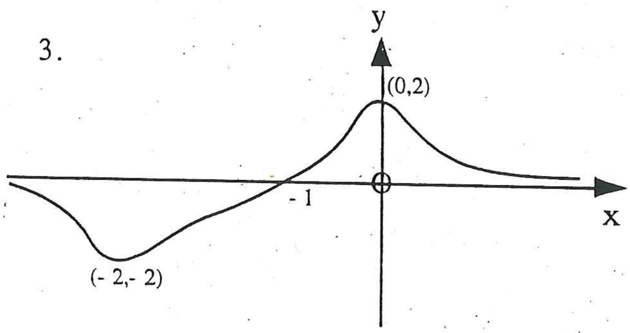
1.



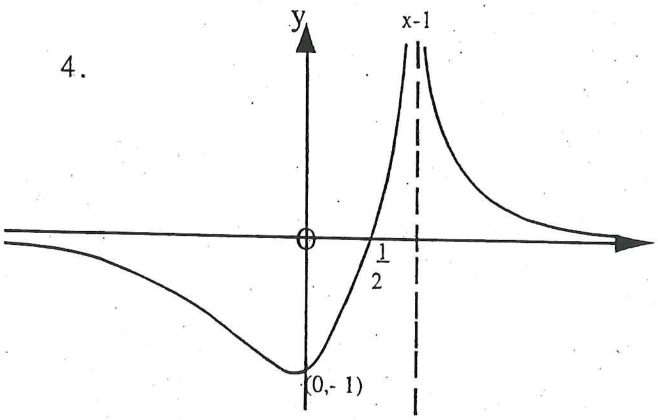
2.



3.



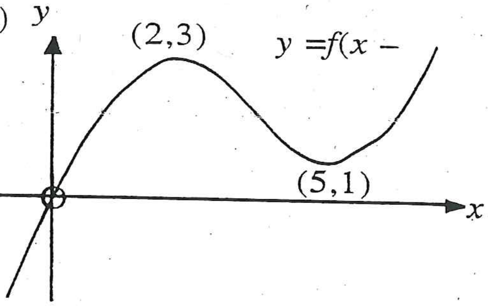
4.



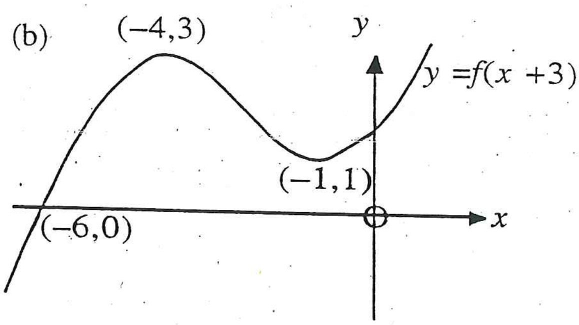
Exercise 5 Page 68

1.	(a)	$^1/_2x$	(b)	$2 - x$	(c)	$^2/_x$
	(d)	$\log_2 x$	(e)	$1 - x$	(f)	$10^x + 2$
2.	(a)	p/3	(b)	p/6	(c)	p/4
	(d)	p/6	(e)	–p/6	(f)	p/6

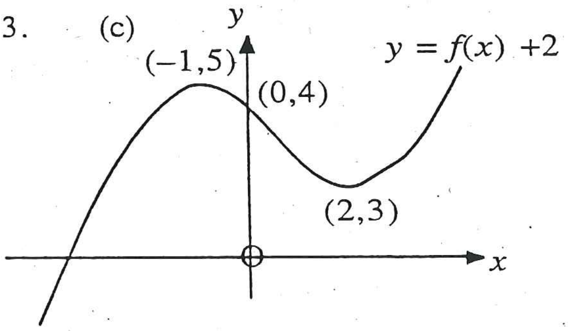
3. (a)



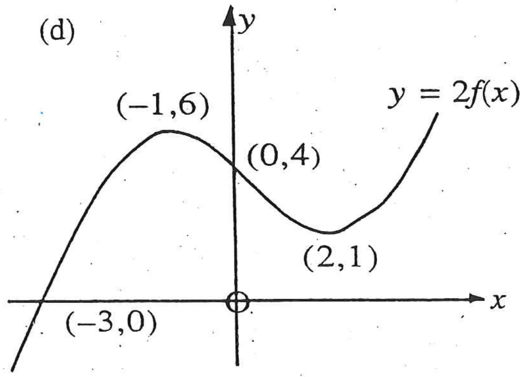
(b)

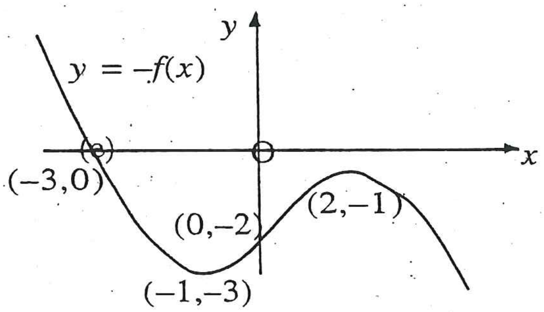


3. (c)

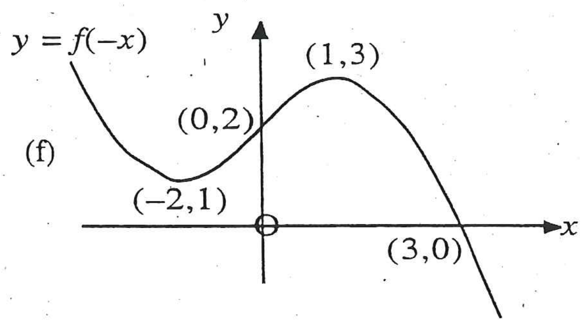

y
(−1,5)
(0,4)
y = f(x) +2
(2,3)
x

(d)

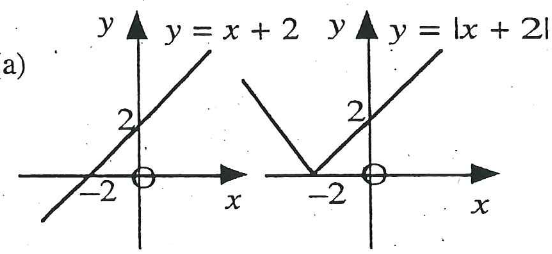

y
(−1,6)
y = 2f(x)
(0,4)
(2,1)
x
(−3,0)


y
y = −f(x)
x
(−3,0)
(0,−2)
(2,−1)
(−1,−3)

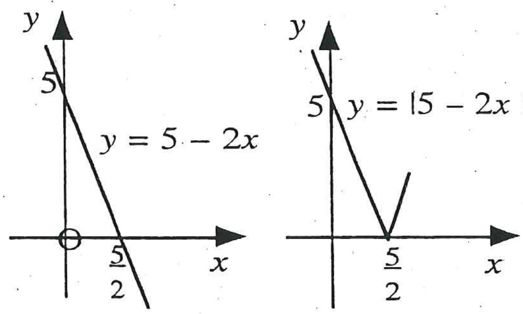
(f)


y = f(−x)
y
(1,3)
(0,2)
(−2,1)
x
(3,0)

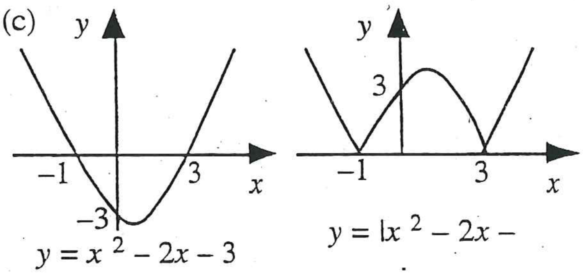
4. (a)


y
y = x + 2
y
y = |x + 2|
2
−2
x
2
−2
x

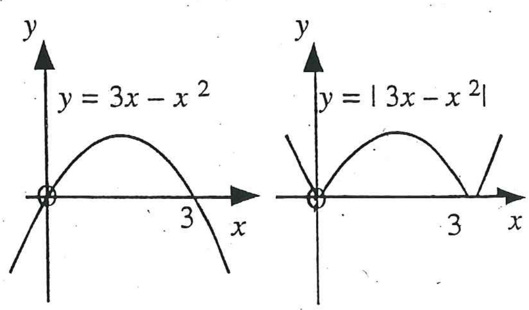
(b)


y
5
y = 5 − 2x
5/2
x
y
5
y = |5 − 2x |
5/2
x

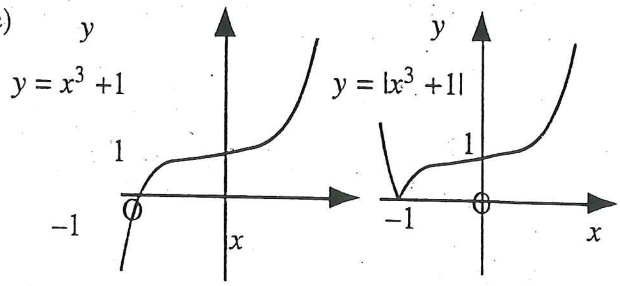
(c)


y
−1
3
x
−3
y = x² − 2x − 3
y
3
−1
3
x
y = |x² − 2x −

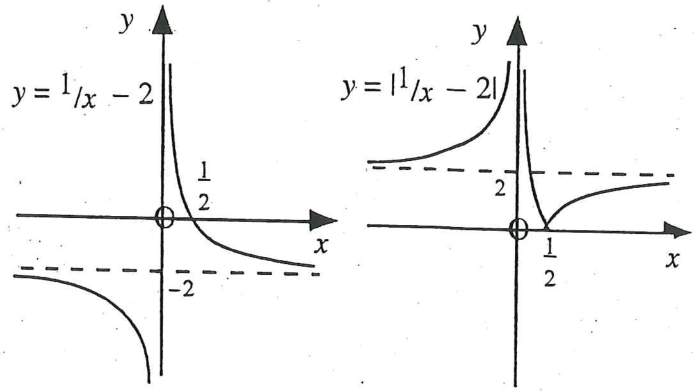
(d)


y
y = 3x − x²
3
x
y
y = | 3x − x²|
3
x

(e)



(f)



5. (a) neither (b) even (c) odd
(d) odd (e) even (f) even