Higher

Mathematics

Integration

Contents

CfE Edition

Integration

1 Indefinite Integrals

In **integration**, our aim is to "undo" the process of differentiation. Later we will see that integration is a useful tool for evaluating areas and solving a special type of equation.

We have already seen how to differentiate polynomials, so we will now look at how to undo this process. The basic technique is:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1, \; c \text{ is the constant of integration.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add the constant of integration (c).

EXAMPLES

1. Find $\int x^2 \, dx$.

$$\int x^2 \, dx = \frac{x^3}{3} + c = \tfrac{1}{3}x^3 + c.$$

2. Find $\int x^{-3} \, dx$.

$$\int x^{-3} \, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$$

3. Find $\int x^{\frac{5}{4}} \, dx$.

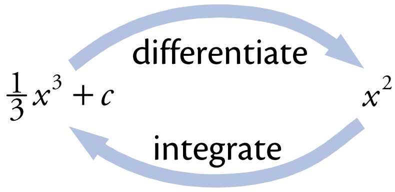
$$\int x^{\frac{5}{4}} \, dx = \frac{x^{\frac{9}{4}}}{\frac{9}{4}} + c = \tfrac{4}{9}x^{\frac{9}{4}} + c.$$

- We use the symbol $\int$ for integration.
- The $\int$ must be used with "dx" in the examples above, to indicate that we are integrating with respect to x.
- The constant of integration is included to represent any constant term in the original expression, since this would have been zeroed by differentiation.
- Integrals with a constant of integration are called **indefinite integrals**.

Checking the answer

Since integration and differentiation are reverse processes, if we differentiate our answer we should get back to what we started with.

For example, if we differentiate our answer to Example 1 above, we do get back to the expression we started with.



Integrating terms with coefficients

The above technique can be extended to:

$$\int ax^n \, dx = a\int x^n \, dx = \frac{ax^{n+1}}{n+1} + c \qquad n \neq -1, \ a \text{ is a constant.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add on c.

EXAMPLES

4. Find $\int 6x^3 \, dx$.

$$\begin{aligned}\int 6x^3 \, dx &= \frac{6x^4}{4} + c \\ &= \tfrac{3}{2}x^4 + c.\end{aligned}$$

5. Find $\int 4x^{-\frac{3}{2}} \, dx$.

$$\begin{aligned}\int 4x^{-\frac{3}{2}} \, dx &= \frac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + c \\ &= -8x^{-\frac{1}{2}} + c \\ &= -\frac{8}{\sqrt{x}} + c.\end{aligned}$$

Note

It can be easy to confuse integration and differentiation, so remember:

$$\int x \, dx = \tfrac{1}{2}x^2 + c \qquad\qquad \int k \, dx = kx + c.$$

Other variables

Just as with differentiation, we can integrate with respect to any variable.

EXAMPLES

6. Find $\int 2p^{-5}\, dp$.

$$\begin{aligned}\int 2p^{-5}\, dp &= \frac{2p^{-4}}{-4} + c \\ &= -\frac{1}{2p^4} + c.\end{aligned}$$

Note
dp tells us to integrate with respect to p.

7. Find $\int p\, dx$.

$$\begin{aligned}&\int p\, dx \\ &= px + c.\end{aligned}$$

Note
Since we are integrating with respect to x, we treat p as a constant.

Integrating several terms

The following rule is used to integrate an expression with several terms:

$$\int \left(f(x) + g(x)\right) dx = \int f(x)\, dx + \int g(x)\, dx.$$

Stated simply: integrate each term separately.

EXAMPLES

8. Find $\int \left(3x^2 - 2x^{\frac{1}{2}}\right) dx$.

$$\begin{aligned}\int \left(3x^2 - 2x^{\frac{1}{2}}\right) dx &= \frac{3x^3}{3} - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= x^3 - \frac{4x^{\frac{3}{2}}}{3} + c \\ &= x^3 - \tfrac{4}{3}\sqrt{x^3} + c.\end{aligned}$$

9. Find $\int \left(4x^{-\frac{5}{8}} + 3x + 7\right) dx$.

$$\begin{aligned}\int \left(4x^{-\frac{5}{8}} + 3x + 7\right) dx &= \frac{4x^{\frac{3}{8}}}{\frac{3}{8}} + \frac{3x^2}{2} + 7x + c \\ &= \tfrac{8}{3} \times 4x^{\frac{3}{8}} + \tfrac{3}{2}x^2 + 7x + c \\ &= \tfrac{32}{3}x^{\frac{3}{8}} + \tfrac{3}{2}x^2 + 7x + c.\end{aligned}$$

2 Preparing to Integrate

As with differentiation, it is important that before integrating, all brackets are multiplied out, and there are no fractions with an x term in the denominator (bottom line), for example:

$$\frac{1}{x^3} = x^{-3} \qquad \frac{3}{x^2} = 3x^{-2} \qquad \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \qquad \frac{1}{4x^5} = \frac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2} = \frac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Find $\int \frac{dx}{x^2}$ for $x \neq 0$.

$\int \frac{dx}{x^2}$ is just a short way of writing $\int \frac{1}{x^2}\, dx$, so:

$$\int \frac{dx}{x^2} = \int \frac{1}{x^2}\, dx = \int x^{-2} dx$$
$$= \frac{x^{-1}}{-1} + c$$
$$= -\frac{1}{x} + c.$$

2. Find $\int \frac{dx}{\sqrt{x}}$ for $x > 0$.

$$\int \frac{dx}{\sqrt{x}} = \int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}} dx$$
$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 2\sqrt{x} + c.$$

3. Find $\int \frac{7}{2p^2}\, dp$ where $p \neq 0$.

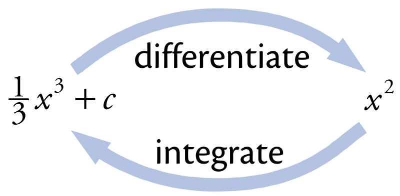
$$\int \frac{7}{2p^2}\, dp = \int \tfrac{7}{2} p^{-2}\, dp$$
$$= \tfrac{7}{2} \times \frac{p^{-1}}{-1} + c$$
$$= -\frac{7}{2p} + c.$$

4. Find $\displaystyle\int \frac{3x^5 - 5x}{4}\, dx$.

$$\begin{aligned}\int \frac{3x^5 - 5x}{4}\, dx &= \int \left(\tfrac{3}{4}x^5 - \tfrac{5}{4}x\right) dx \\ &= \frac{3x^6}{4\times 6} - \frac{5x^2}{4\times 2} + c \\ &= \tfrac{3}{24}x^6 - \tfrac{5}{8}x^2 + c \\ &= \tfrac{1}{8}x^6 - \tfrac{5}{8}x^2 + c.\end{aligned}$$

3 Differential Equations

A **differential equation** is an equation involving derivatives, e.g. $\dfrac{dy}{dx} = x^2$.

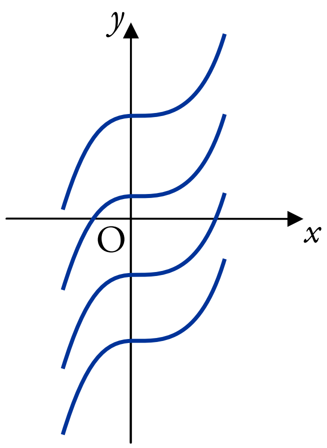


A solution of a differential equation is an expression for the original function; in this case $y = \frac{1}{3}x^3 + c$ is a solution.

In general, we obtain solutions using integration:

$$y = \int \frac{dy}{dx}\, dx \qquad \text{or} \qquad f(x) = \int f'(x)\, dx.$$

This will result in a **general solution** since we can choose the value of c.



The general solution corresponds to a "family" of curves, each with a different value for c.

The graph to the left illustrates some of the curves $y = \frac{1}{3}x^3 + c$ with different values of c.

If we have additional information about the function (such as a point its graph passes through), we can find the value of c and obtain a **particular solution**.

EXAMPLES

1. The graph of $y = f(x)$ passes through the point $(3, -4)$.

   If $\frac{dy}{dx} = x^2 - 5$, express y in terms of x.

$$\begin{aligned} y &= \int \frac{dy}{dx}\, dx \\ &= \int (x^2 - 5)\, dx \\ &= \tfrac{1}{3}x^3 - 5x + c. \end{aligned}$$

We know that when $x = 3$, $y = -4$ so we can find c:

$$\begin{aligned} y &= \tfrac{1}{3}x^3 - 5x + c \\ -4 &= \tfrac{1}{3}(3)^3 - 5(3) + c \\ -4 &= 9 - 15 + c \\ c &= 2 \end{aligned}$$

So $y = \tfrac{1}{3}x^3 - 5x + 2$.

2. The function f, defined on a suitable domain, is such that

   $f'(x) = x^2 + \frac{1}{x^2} + \tfrac{2}{3}$.

   Given that $f(1) = 4$, find a formula for $f(x)$ in terms of x.

$$\begin{aligned} f(x) &= \int f'(x)\, dx \\ &= \int \left(x^2 + \frac{1}{x^2} + \tfrac{2}{3} \right) dx \\ &= \int \left(x^2 + x^{-2} + \tfrac{2}{3} \right) dx \\ &= \tfrac{1}{3}x^3 - x^{-1} + \tfrac{2}{3}x + c \\ &= \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + c. \end{aligned}$$

We know that $f(1) = 4$, so we can find c:

$$\begin{aligned} f(x) &= \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + c \\ 4 &= \tfrac{1}{3}(1)^3 - \tfrac{1}{1} + \tfrac{2}{3}(1) + c \\ 4 &= \tfrac{1}{3} - 1 + \tfrac{2}{3} + c \\ c &= 4. \end{aligned}$$

So $f(x) = \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + 4$.

4 Definite Integrals

If $F(x)$ is an integral of $f(x)$, then we define:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

where a and b are called the **limits** of the integral.

Stated simply:

- Work out the integral as normal, leaving out the constant of integration.
- Evaluate the integral for $x = b$ (the upper limit value).
- Evaluate the integral for $x = a$ (the lower limit value).
- Subtract the lower limit value from the upper limit value.

Since there is no constant of integration and we are calculating a numerical value, this is called a **definite integral**.

EXAMPLES

1. Find $\int_1^3 5x^2\,dx$.

$$\begin{aligned}\int_1^3 5x^2\,dx &= \left[\frac{5x^3}{3}\right]_1^3 \\ &= \left(\frac{5(3)^3}{3}\right) - \left(\frac{5(1)^3}{3}\right) \\ &= 5\times 3^2 - \tfrac{5}{3} \\ &= 45 - \tfrac{5}{3} = 43\tfrac{1}{3}.\end{aligned}$$

2. Find $\int_0^2 \left(x^3 + 3x^2\right) dx$.

$$\begin{aligned}\int_0^2 \left(x^3 + 3x^2\right) dx &= \left[\frac{x^4}{4} + \frac{3x^3}{3}\right]_0^2 \\ &= \left[\frac{x^4}{4} + x^3\right]_0^2 \\ &= \left(\frac{2^4}{4} + 2^3\right) - \left(\frac{0^4}{4} + 0^3\right) \\ &= \tfrac{16}{4} + 8 - 0 \\ &= 4 + 8 = 12.\end{aligned}$$

3. Find $\displaystyle\int_{-1}^{4} \frac{4}{x^3}\,dx$.

$$\begin{aligned}\int_{-1}^{4} \frac{4}{x^3}\,dx &= \int_{-1}^{4} 4x^{-3}\,dx \\ &= \left[\frac{4x^{-2}}{-2}\right]_{-1}^{4} \\ &= \left[-\frac{2}{x^2}\right]_{-1}^{4} \\ &= \left(-\frac{2}{4^2}\right)-\left(-\frac{2}{(-1)^2}\right) \\ &= -\tfrac{2}{16}+2 = 1\tfrac{7}{8}.\end{aligned}$$

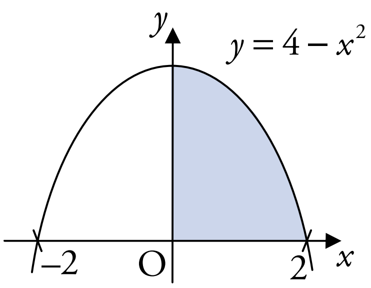
5 Geometric Interpretation of Integration

We will now consider the meaning of integration in the context of areas.

Consider $\int_0^2 \left(4-x^2\right)dx = \left[4x-\tfrac{1}{3}x^3\right]_0^2$

$$\begin{aligned} &= \left(8-\tfrac{8}{3}\right)-0 \\ &= 5\tfrac{1}{3}.\end{aligned}$$

On the graph of $y = 4-x^2$:



The shaded area is given by $\int_0^2 \left(4-x^2\right)dx$.

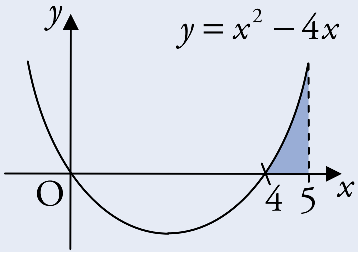
Therefore the shaded area is $5\frac{1}{3}$ square units.

In general, the area enclosed by the graph $y = f(x)$ and the x-axis, between $x = a$ and $x = b$, is given by

$$\int_a^b f(x)\,dx.$$

EXAMPLE

1. The graph of $y = x^2 - 4x$ is shown below. Calculate the shaded area.

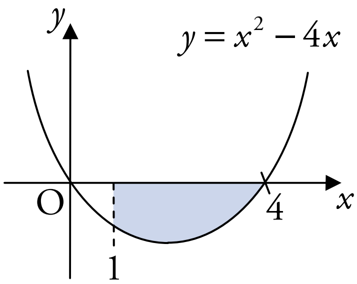


$$\int_4^5 \left(x^2 - 4x\right) dx = \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_4^5$$
$$= \left(\frac{5^3}{3} - 2(5)^2\right) - \left(\frac{4^3}{3} - 2(4)^2\right)$$
$$= \tfrac{125}{3} - 50 - \tfrac{64}{3} + 32$$
$$= \tfrac{61}{3} - 18$$
$$= 2\tfrac{1}{3}.$$

So the shaded area is $2\frac{1}{3}$ square units.

Areas below the *x*-axis

Care needs to be taken if part or all of the area lies below the *x*-axis. For example if we look at the graph of $y = x^2 - 4$:



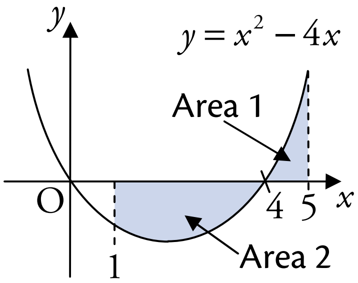
The shaded area is given by

$$\int_1^4 \left(x^2 - 4x\right) dx = \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^4$$
$$= \left(\frac{4^3}{3} - 2(4)^2\right) - \left(\tfrac{1}{3} - 2\right)$$
$$= \tfrac{64}{3} - 32 - \tfrac{1}{3} + 2$$
$$= \tfrac{63}{3} - 30 = 21 - 30 = -9.$$

In this case, the negative indicates that the area is below the *x*-axis, as can be seen from the diagram. The area is therefore 9 square units.

Areas above and below the *x*-axis

Consider the graph from the example above, with a different shaded area:



From the examples above, the total shaded area is:

$$\text{Area 1} + \text{Area 2} = 2\tfrac{1}{3} + 9 = 11\tfrac{1}{3} \text{ square units.}$$

Using the method from above, we might try to calculate the shaded area as follows:

$$\begin{aligned}\int_1^5 \left(x^2 - 4x\right) dx &= \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^5 \\ &= \left(\frac{5^3}{3} - 2(5)^2\right) - \left(\tfrac{1}{3} - 2\right) \\ &= \tfrac{125}{3} - 50 - \tfrac{1}{3} + 2 \\ &= \tfrac{124}{3} - 48 = -6\tfrac{2}{3}.\end{aligned}$$

Clearly this shaded area is not $6\tfrac{2}{3}$ square units since we already found it to be $11\tfrac{1}{3}$ square units. This problem arises because Area 1 is above the *x*-axis, while Area 2 is below.

To find the true area, we needed to evaluate two integrals:

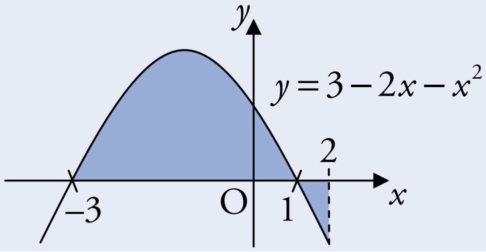
$$\int_1^4 \left(x^2 - 4x\right) dx \qquad \text{and} \qquad \int_4^5 \left(x^2 - 4x\right) dx.$$

We then found the total shaded area by adding the two areas together.

We must take care to do this whenever the area is split up in this way.

EXAMPLES

2. Calculate the shaded area shown in the diagram below.



To calculate the area from $x = -3$ to $x = 1$:

$$\begin{aligned}\int_{-3}^{1} \left(3 - 2x - x^2\right) dx &= \left[3x - \frac{2x^2}{2} - \frac{x^3}{3}\right]_{-3}^{1} \\ &= \left[3x - x^2 - \tfrac{1}{3}x^3\right]_{-3}^{1} \\ &= \left(3(1) - (1)^2 - \tfrac{1}{3}(1)^3\right) - \left(3(-3) - (-3)^2 - \tfrac{1}{3}(-3)^3\right) \\ &= \left(3 - 1 - \tfrac{1}{3}\right) - (-9 - 9 + 9) \\ &= 3 - 1 - \tfrac{1}{3} + 9 \\ &= 10\tfrac{2}{3}\end{aligned}$$

So the area is $10\tfrac{2}{3}$ square units.

We have already integrated the equation of the curve, so we can just substitute in new limits to work out the area from $x = 1$ to $x = 2$:

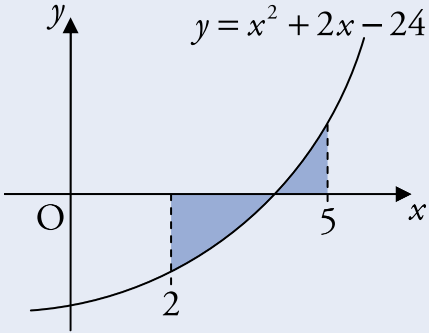
$$\begin{aligned}\int_{1}^{2} \left(3 - 2x - x^2\right) dx &= \left[3x - x^2 - \tfrac{1}{3}x^3\right]_{1}^{2} \\ &= \left(3(2) - (2)^2 - \tfrac{1}{3}(2)^3\right) - \left(3(1) - (1)^2 - \tfrac{1}{3}(1)^3\right) \\ &= \left(6 - 4 - \tfrac{8}{3}\right) - \left(3 - 1 - \tfrac{1}{3}\right) \\ &= 2 - \tfrac{8}{3} - 2 + \tfrac{1}{3} \\ &= -2\tfrac{1}{3}.\end{aligned}$$

So the area is $2\tfrac{1}{3}$ square units.

Remember
The negative sign just indicates that the area lies below the axis.

So the total shaded area is $10\tfrac{2}{3} + 2\tfrac{1}{3} = 13$ square units.

3. Calculate the shaded area shown in the diagram below.



First, we need to calculate the root between $x = 2$ and $x = 5$:

$$x^2 + 2x - 24 = 0$$
$$(x-4)(x+6) = 0$$
$$x = 4 \text{ or } x = -6.$$

So the root is $x = 4$

To calculate the area from $x = 2$ to $x = 4$:

$$\begin{aligned}\int_2^4 \left(x^2 + 2x - 24\right) dx &= \left[\frac{x^3}{3} + \frac{2x^2}{2} - 24x\right]_2^4 \\ &= \left[\tfrac{1}{3}x^3 + x^2 - 24x\right]_2^4 \\ &= \left(\tfrac{1}{3}(4)^3 + (4)^2 - 24(4)\right) - \left(\tfrac{1}{3}(2)^3 + (2)^2 - 24(2)\right) \\ &= \left(\tfrac{64}{3} + 16 - 96\right) - \left(\tfrac{8}{3} + 4 - 48\right) \\ &= \tfrac{56}{3} - 36 \\ &= -17\tfrac{1}{3}\end{aligned}$$

So the area is $17\frac{1}{3}$ square units.

To calculate the area from $x = 4$ to $x = 5$:

$$\begin{aligned}\int_4^5 \left(x^2 + 2x - 24\right) dx &= \left[\tfrac{1}{3}x^3 + x^2 - 24x\right]_4^5 \\ &= \left(\tfrac{1}{3}(5)^3 + (5)^2 - 24(5)\right) - \left(\tfrac{1}{3}(4)^3 + (4)^2 - 24(4)\right) \\ &= \left(\tfrac{125}{3} + 25 - 120\right) - \left(\tfrac{64}{3} + 16 - 96\right) \\ &= \tfrac{61}{3} - 15 \\ &= 5\tfrac{1}{3}\end{aligned}$$

So the area is $5\frac{1}{3}$ square units.

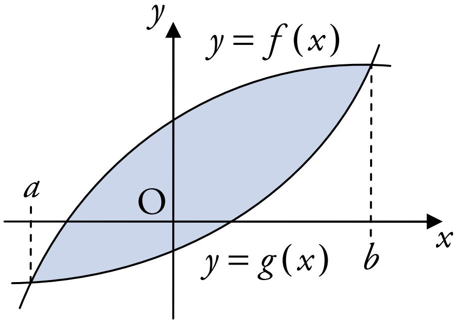
So the total shaded area is $17\frac{1}{3} + 5\frac{1}{3} = 22\frac{2}{3}$ square units.

6 Areas between Curves

The area between two curves between $x = a$ and $x = b$ is calculated as:

$$\int_a^b (\text{upper curve} - \text{lower curve})\, dx \text{ square units.}$$

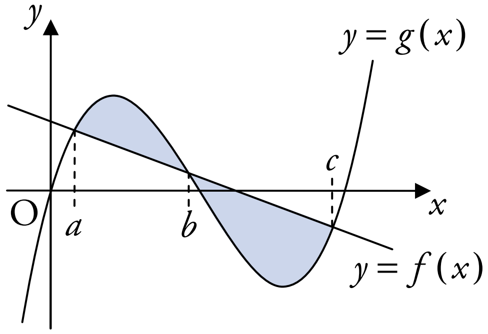
So for the shaded area shown below:



The area is $\int_a^b (f(x) - g(x))\, dx$ square units.

When dealing with areas between curves, areas above and below the x-axis do not need to be calculated separately.

However, care must be taken with more complicated curves, as these may give rise to more than one closed area. These areas must be evaluated separately. For example:



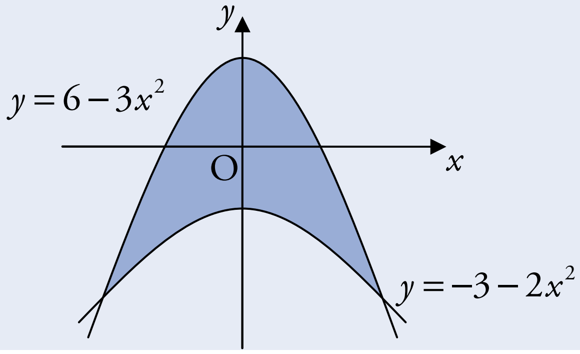
In this case we apply $\int_a^b (\text{upper curve} - \text{lower curve})\, dx$ to each area.

So the shaded area is given by:

$$\int_a^b (g(x) - f(x))\, dx + \int_b^c (f(x) - g(x))\, dx.$$

EXAMPLES

1. Calculate the shaded area enclosed by the curves with equations $y = 6 - 3x^2$ and $y = -3 - 2x^2$.



To work out the points of intersection, equate the curves:

$$\begin{aligned} 6 - 3x^2 &= -3 - 2x^2 \\ 6 + 3 - 3x^2 + 2x^2 &= 0 \\ 9 - x^2 &= 0 \\ (3 + x)(3 - x) &= 0 \\ x = -3 \quad &\text{or} \quad x = 3. \end{aligned}$$

Set up the integral and simplify:

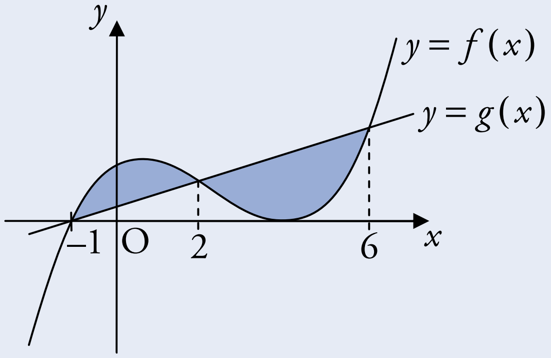
$$\begin{aligned} &\int_{-3}^{3} (\text{upper curve} - \text{lower curve})\, dx \\ &= \int_{-3}^{3} \left((6 - 3x^2) - (-3 - 2x^2) \right) dx \\ &= \int_{-3}^{3} \left(6 - 3x^2 + 3 + 2x^2 \right) dx \\ &= \int_{-3}^{3} \left(9 - x^2 \right) dx. \end{aligned}$$

Carry out integration:

$$\begin{aligned} \int_{-3}^{3} (9 - x^2)\, dx &= \left[9x - \frac{x^3}{3} \right]_{-3}^{3} \\ &= \left(9(3) - \frac{(3)^3}{3} \right) - \left(9(-3) - \frac{(-3)^3}{3} \right) \\ &= \left(27 - \tfrac{27}{3} \right) - \left(-27 + \tfrac{27}{3} \right) \\ &= 27 - 9 + 27 - 9 \\ &= 36. \end{aligned}$$

Therefore the shaded area is 36 square units.

2. Two functions are defined for $x \in \mathbb{R}$ by $f(x) = x^3 - 7x^2 + 8x + 16$ and $g(x) = 4x + 4$. The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



Calculate the shaded area.

Since the shaded area is in two parts, we apply $\int_a^b (\text{upper} - \text{lower})\, dx$ twice.

Area from $x = -1$ to $x = 2$:

$$\int_{-1}^{2} (\text{upper} - \text{lower})\, dx$$
$$= \int_{-1}^{2} \left(x^3 - 7x^2 + 8x + 16 - (4x + 4) \right) dx$$
$$= \int_{-1}^{2} \left(x^3 - 7x^2 + 4x + 12 \right) dx$$
$$= \left[\frac{x^4}{4} - \frac{7x^3}{3} + \frac{4x^2}{2} + 12x \right]_{-1}^{2}$$
$$= \left(\frac{2^4}{4} - \frac{7 \times 2^3}{3} + 2 \times 2^2 + 12 \times 2 \right) - \left(\frac{(-1)^4}{4} - \frac{7(-1)^3}{3} + 2(-1)^2 + 12(-1) \right)$$
$$= \left(4 - \tfrac{56}{3} + 8 + 24 \right) - \left(\tfrac{1}{4} - \tfrac{7}{3} + 2 - 12 \right)$$
$$= \frac{99}{4}$$
$$= 24\tfrac{3}{4}.$$

Note
The curve is at the top of this area.

So the area is $24\tfrac{3}{4}$ square units.

Area from $x = 2$ to $x = 6$:

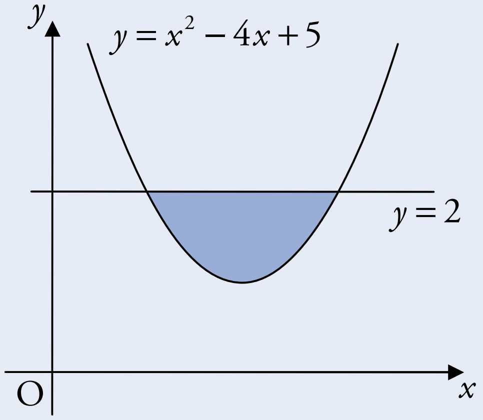
$$\int_2^6 (\text{upper} - \text{lower})\, dx$$
$$= \int_2^6 \left(4x - 4 - \left(x^3 - 7x^2 + 8x + 16\right)\right) dx$$
$$= \int_2^6 \left(-x^3 + 7x^2 - 4x - 12\right) dx$$
$$= \left[-\frac{x^4}{4} + \frac{7x^3}{3} - \frac{4x^2}{2} - 12x\right]_2^6$$
$$= \left(-\frac{6^4}{4} + \frac{7\times 6^3}{3} - \frac{4\times 6^2}{2} - 12\times 6\right) - \left(-\frac{2^4}{4} + \frac{7\times 2^3}{3} - \frac{4\times 2^2}{2} - 12\times 2\right)$$
$$= (-324 + 504 - 72 - 72) - \left(-4 + \tfrac{56}{3} - 8 - 24\right)$$
$$= \tfrac{160}{3}$$
$$= 53\tfrac{1}{3}.$$

Note
The straight line is at the top of this area.

So the area is $53\frac{1}{3}$ square units.

So the total shaded area is $24\frac{3}{4} + 53\frac{1}{3} = 78\frac{1}{12}$ square units.

3. A trough is 2 metres long. A cross-section of the trough is shown below.



The cross-section is part of the parabola with equation $y = x^2 - 4x + 5$.

Find the volume of the trough.

To work out the points of intersection, equate the curve and the line:

$$x^2 - 4x + 5 = 2$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0 \quad \text{so} \quad x = 1 \text{ or } x = 3.$$

Set up the integral and integrate:

$$\begin{aligned}
\int_1^3 (\text{upper} - \text{lower})\, dx &= \int_1^3 \left(2 - \left(x^2 - 4x + 5\right)\right) dx \\
&= \int_1^3 \left(-x^2 + 4x - 3\right) dx \\
&= \left[-\frac{x^3}{3} + \frac{4x^2}{2} - 3x\right]_1^3 \\
&= \left(-\frac{(3)^3}{3} + 2(3)^2 - 3(3)\right) - \left(-\frac{(1)^3}{3} + 2(1)^2 - 3(1)\right) \\
&= (-9 + 18 - 9) - \left(-\tfrac{1}{3} + 2 - 3\right) \\
&= 0 + \tfrac{1}{3} - 2 + 3 \\
&= \tfrac{4}{3} \\
&= 1\tfrac{1}{3}.
\end{aligned}$$

Therefore the shaded area is $1\frac{1}{3}$ square units.

$$\begin{aligned}
\text{Volume} &= \text{cross-sectional area} \times \text{length} \\
&= \tfrac{4}{3} \times 2 \\
&= \tfrac{8}{3} \quad = 2\tfrac{2}{3}.
\end{aligned}$$

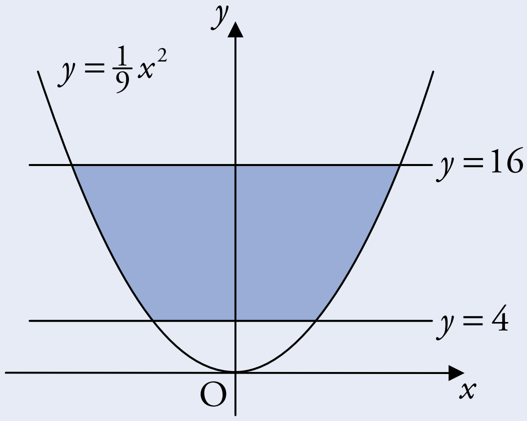
Therefore the volume of the trough is $2\frac{2}{3}$ cubic units.

7 Integrating along the *y*-axis

For some problems, it may be easier to find a shaded area by integrating with respect to y rather than x.

EXAMPLE

The curve with equation $y=\frac{1}{9}x^2$ is shown in the diagram below.



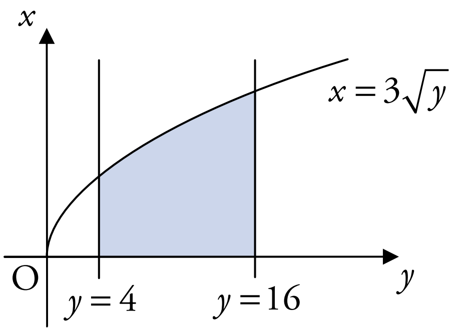
Calculate the shaded area which lies between $y=4$ and $y=16$.

We have $y=\frac{1}{9}x^2$

$$9y=x^2$$

$$x=\pm\sqrt{9y}$$

$$x=\pm 3\sqrt{y}.$$



The shaded area in the diagram to the right is given by:

$$\int_4^{16} 3\sqrt{y}\,dy=\int_4^{16} 3y^{\frac{1}{2}}\,dy$$

$$=\left[\frac{3y^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^{16}$$

$$=\left[2\sqrt{y}^3\right]_4^{16}$$

$$=2\sqrt{16}^3-2\sqrt{4}^3$$

$$=2\times 64-2\times 8$$

$$=112.$$

Since this is half of the required area, the total shaded area is 224 square units.

8 Integrating sinx and cosx

We know the derivatives of $\sin x$ and $\cos x$, so it follows that the integrals are:

$$\int \cos x \, dx = \sin x + c, \qquad \int \sin x \, dx = -\cos x + c.$$

Again, these results only hold if x is measured in radians.

EXAMPLES

1. Find $\int (5\sin x + 2\cos x)\, dx$.

$$\int (5\sin x + 2\cos x)\, dx = -5\cos x + 2\sin x + c.$$

2. Find $\int_0^{\frac{\pi}{4}} (4\cos x + 2\sin x)\, dx$.

$$\begin{aligned}\int_0^{\frac{\pi}{4}} 4\cos x + 2\sin x \, dx &= \left[4\sin x - 2\cos x\right]_0^{\frac{\pi}{4}} \\ &= \left[4\sin\left(\tfrac{\pi}{4}\right) - 2\cos\left(\tfrac{\pi}{4}\right)\right] - \left[4\sin 0 - 2\cos 0\right] \\ &= \left[\left(4\times\tfrac{1}{\sqrt{2}}\right) - \left(2\times\tfrac{1}{\sqrt{2}}\right)\right] - [-2] \\ &= \tfrac{4}{\sqrt{2}} - \tfrac{2}{\sqrt{2}} + 2 \\ &= \left(\tfrac{2}{\sqrt{2}}\times\tfrac{\sqrt{2}}{\sqrt{2}}\right) + 2 \\ &= \sqrt{2} + 2.\end{aligned}$$

Note
It is good practice to rationalise the denominator.

3. Find the value of $\int_0^4 \frac{1}{2}\sin x \, dx$.

$$\begin{aligned}\int_0^4 \tfrac{1}{2}\sin x \, dx &= \left[-\tfrac{1}{2}\cos x\right]_0^4 \\ &= -\tfrac{1}{2}\cos(4) + \tfrac{1}{2}\cos(0) \\ &= \tfrac{1}{2}(0{\cdot}654 + 1) \\ &= 0{\cdot}827 \text{ (to 3 d.p.)}\end{aligned}$$

Remember
We must use radians when integrating of differentiating trigonometric functions.

9 A Special Integral

The method for integrating an expression of the form $(ax+b)^n$ is:

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \qquad \text{where } a \neq 0 \text{ and } n \neq -1.$$

Stated simply: raise the power (n) by one, divide by the new power and also divide by the derivative of the bracket ($a(n+1)$), add c.

EXAMPLES

1. Find $\int (x+4)^7 \, dx$.

$$\int (x+4)^7 \, dx = \frac{(x+4)^8}{8\times 1} + c$$
$$= \frac{(x+4)^8}{8} + c.$$

2. Find $\int (2x+3)^2 \, dx$.

$$\int (2x+3)^2 \, dx = \frac{(2x+3)^3}{3\times 2} + c$$
$$= \frac{(2x+3)^3}{6} + c.$$

3. Find $\int \frac{1}{\sqrt[3]{5x+9}} \, dx$ where $x \neq -\frac{9}{5}$.

$$\int \frac{1}{\sqrt[3]{5x+9}} \, dx = \int \frac{1}{(5x+9)^{\frac{1}{3}}} \, dx$$
$$= \int (5x+9)^{-\frac{1}{3}} \, dx$$
$$= \frac{(5x+9)^{\frac{2}{3}}}{\frac{2}{3}\times 5} + c$$
$$= \frac{\sqrt[3]{5x+9}^{\,2}}{\frac{10}{3}} + c$$
$$= \frac{3}{10}\sqrt[3]{5x+9}^{\,2} + c.$$

4. Evaluate $\int_0^3 \sqrt{3x+4}\, dx$ where $x \geq -\frac{4}{3}$.

$$\begin{aligned}\int_0^3 \sqrt{3x+4}\, dx &= \int_0^3 (3x+4)^{\frac{1}{2}}\, dx \\ &= \left[\frac{(3x+4)^{\frac{3}{2}}}{\frac{3}{2}\times 3}\right]_0^3 \\ &= \left[\frac{2\sqrt{(3x+4)^3}}{9}\right]_0^3 \\ &= \left[\frac{2\sqrt{(3(3)+4)^3}}{9}\right] - \left[\frac{2\sqrt{(3(0)+4)^3}}{9}\right] \\ &= \frac{2\sqrt{13^3}}{9} - \frac{2\sqrt{4^3}}{9} \\ &= \frac{2}{9}\left(\sqrt{13^3} - 8\right) \qquad \text{(or } 8{\cdot}638 \text{ to 3 d.p.).}\end{aligned}$$

Note
Changing powers back to roots here makes it easier to evaluate the two brackets.

Remember
To evaluate $\sqrt{4^3}$, it is easier to work out $\sqrt{4}$ first.

Warning

Make sure you don't confuse differentiation and integration – this could lose you a lot of marks in the exam.

Remember the following rules for differentiation and integrating expressions of the form $(ax+b)^n$:

$$\frac{d}{dx}\left[(ax+b)^n\right] = an(ax+b)^{n-1},$$

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

These rules will *not* be given in the exam.

Using Differentiation to Integrate

Recall that integration is the process of undoing differentiation. So if we differentiate $f(x)$ to get $g(x)$ then we know that $\int g(x)\,dx = f(x)+c$.

EXAMPLES

5. (a) Differentiate $y=\dfrac{5}{(3x-1)^4}$ with respect to x.

(b) Hence, or otherwise, find $\displaystyle\int \frac{1}{(3x-1)^5}\,dx$.

(a) $$y=\frac{5}{(3x-1)^4}=5(3x-1)^{-4}$$
$$\frac{dy}{dx}=5\times3\times(-4)(3x-1)^{-5}$$
$$=-\frac{60}{(3x-1)^5}.$$

(b) From part (a) we know $\displaystyle\int -\frac{60}{(3x-1)^5}\,dx=\frac{5}{(3x-1)^4}+c$. So:

$$-60\int\frac{1}{(3x-1)^5}\,dx=\frac{5}{(3x-1)^4}+c$$
$$\int\frac{1}{(3x-1)^5}\,dx=-\frac{1}{60}\left(\frac{5}{(3x-1)^4}+c\right)$$
$$=-\frac{1}{12(3x-1)^4}+c_1$$ where c_1 is some constant.

Note
We could also have used the special integral to obtain this answer.

6. (a) Differentiate $y=\dfrac{1}{(x^3-1)^5}$ with respect to x.

(b) Hence, find $\displaystyle\int\frac{x^2}{(x^3-1)^6}\,dx$.

(a) $$y=\frac{1}{(x^3-1)^5}=(x^3-1)^{-5}$$
$$\frac{dy}{dx}=-5(x^3-1)^{-6}\times3x^2$$
$$=-\frac{15x^2}{(x^3-1)^6}.$$

(b) From part (a) we know $\int -\frac{15x^2}{(x^3-1)^6}\, dx = \frac{1}{(x^3-1)^5} + c$. So:

$$-15\int \frac{x^2}{(x^3-1)^6}\, dx = \frac{1}{(x^3-1)^5} + c$$

$$\int \frac{x^2}{(x^3-1)^6}\, dx = -\tfrac{1}{15}\left(\frac{1}{(x^3-1)^5} + c\right)$$

$$= -\frac{1}{15(x^3-1)^5} + c_2 \qquad \text{where } c_2 \text{ is some constant.}$$

Note
In this case, the special integral cannot be used.

10 Integrating sin(*ax* + *b*) and cos(*ax* + *b*)

Since we know the derivatives of $\sin(ax+b)$ and $\cos(ax+b)$, it follows that the integrals are:

$$\int \cos(ax+b)\; dx = \tfrac{1}{a}\sin(ax+b) + c,$$

$$\int \sin(ax+b)\; dx = -\tfrac{1}{a}\cos(ax+b) + c.$$

These are given in the exam.

EXAMPLES

1. Find $\int \sin(4x+1)\; dx$.

$$\int \sin(4x+1)\; dx = -\tfrac{1}{4}\cos(4x+1) + c.$$

2. Find $\int \cos\left(\tfrac{3}{2}x + \tfrac{\pi}{5}\right) dx$.

$$\int \cos\left(\tfrac{3}{2}x + \tfrac{\pi}{5}\right) dx = \tfrac{2}{3}\sin\left(\tfrac{3}{2}x + \tfrac{\pi}{5}\right) + c.$$

3. Find the value of $\int_0^1 \cos(2x-5)\, dx$.

$$\int_0^1 \cos(2x-5)\, dx = \left[\tfrac{1}{2}\sin(2x-5)\right]_0^1$$

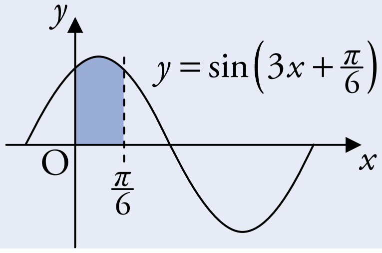
$$= \tfrac{1}{2}\sin(-3) - \tfrac{1}{2}\sin(-5).$$

$$= \tfrac{1}{2}(-0{\cdot}141 - 0{\cdot}959)$$

$$= -0{\cdot}55 \text{ (to 2 d.p.).}$$

Remember
We must use radians when integrating or differentiating trigonometric functions.

4. Find the area enclosed be the graph of $y = \sin\left(3x + \frac{\pi}{6}\right)$, the x-axis, and the lines $x = 0$ and $x = \frac{\pi}{6}$.



$$\begin{aligned}
\int_0^{\frac{\pi}{6}} \sin\left(3x + \tfrac{\pi}{6}\right)\, dx &= \left[-\tfrac{1}{3}\cos\left(3x + \tfrac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}} \\
&= \left[-\tfrac{1}{3}\cos\left(3\left(\tfrac{\pi}{6}\right) + \tfrac{\pi}{6}\right)\right] - \left[-\tfrac{1}{3}\cos\left(3(0) + \tfrac{\pi}{6}\right)\right] \\
&= \left[-\tfrac{1}{3}\cos(90 + 30)^\circ\right] + \left[\tfrac{1}{3}\cos(30)^\circ\right] \\
&= \left[\left(-\tfrac{1}{3}\right)\times\left(-\tfrac{1}{2}\right)\right] + \left[\tfrac{1}{3}\times\tfrac{\sqrt{3}}{2}\right] \\
&= \tfrac{1}{6} + \tfrac{\sqrt{3}}{6} \\
&= \frac{1 + \sqrt{3}}{6}.
\end{aligned}$$

So the area is $\dfrac{1 + \sqrt{3}}{6}$ square units.

5. Find $\int 2\cos\left(\frac{1}{2}x - 3\right)\, dx$.

$$\begin{aligned}
\int 2\cos\left(\tfrac{1}{2}x - 3\right)\, dx &= \frac{2}{\frac{1}{2}}\sin\left(\tfrac{1}{2}x - 3\right) + c \\
&= 4\sin\left(\tfrac{1}{2}x - 3\right) + c
\end{aligned}$$

6. Find $\int 5\cos(2x) + \sin\left(x - \sqrt{3}\right)\, dx$.

$$\int 5\cos(2x) + \sin\left(x - \sqrt{3}\right)\, dx = \tfrac{5}{2}\sin(2x) - \cos\left(x - \sqrt{3}\right) + c$$

7. (a) Differentiate $\dfrac{1}{\cos x}$ with respect to x.

(b) Hence find $\displaystyle\int \frac{\tan x}{\cos x}\,dx$.

(a) $\dfrac{1}{\cos x} = (\cos x)^{-1}$, and $\frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2} \times -\sin x$

$$= \frac{\sin x}{\cos^2 x}.$$

(b) $\dfrac{\tan x}{\cos x} = \dfrac{\frac{\sin x}{\cos x}}{\cos x} = \dfrac{\sin x}{\cos^2 x}.$

From part (a) we know $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = \frac{1}{\cos x} + c$.

Therefore $\displaystyle\int \frac{\tan x}{\cos x}\,dx = \frac{1}{\cos x} + c$.