Advanced Higher Mathematics

Uddingston Grammar

Respecting Learning Caring

Unit 2

Matrix Algebra

Outcome 2 – Use Matrix Algebra

A **matrix (matrices)** is a rectangular array of numbers arranged in rows and columns, the array being enclosed in round (or square) brackets.

e.g. $\begin{pmatrix} x \\ y \end{pmatrix}$ $\begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$ $\begin{pmatrix} 6 & 8 & 10 \\ 3 & 4 & 5 \end{pmatrix}$ $\begin{pmatrix} 4 & -2 & 5 \end{pmatrix}$

2 rows 1 column	2 rows 2 columns	2 rows 3 columns	1 row 3 columns

Each number in the array is called an **entry** or an **element** of the matrix and is identified by first stating the row and then the column in which it appears.

A matrix is often denoted by a capital letter.

The **order** of a matrix is given by stating the number of rows followed by the number of columns.

$$A = \begin{pmatrix} 4 & 6 & 8 \\ 2 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 2 & -6 \end{pmatrix}$$

Matrix A has 2 rows and 3 columns and is said to be **of order 2 x 3**, (reads 2 by 3).

B has the same number of rows as columns and is called a **square matrix of order 2.**

In general, a matrix A, of order (m x n), has m rows and n columns and can be represented as follows, where a_{ij} denotes element in the ith row and jth column.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & - & a_{2n} \\ a_{31} & a_{32} & a_{33} & - & - & - & a_{3n} \\ '' & '' & '' & '' & '' & '' & '' \\ '' & '' & '' & '' & '' & '' & '' \\ '' & '' & '' & '' & '' & '' & '' \\ a_{m1} & a_{m2} & a_{m3} & - & - & - & a_{mn} \end{pmatrix}$$

Equal Matrices :- Two matrices A and B are equal when :-

(a) they are of the same order **and** (b) their corresponding elements are equal.

The Zero Matrix is a matrix whose **elements are all zero.**

Transpose of a Matrix -: A new matrix can be formed from A by writing row 1 as column 1, row 2 as column 2, row 3 as column 3, etc.

This new matrix is called the **Transpose** of A, and is denoted by A′ (reads as "A dashed" or A transpose).

e.g. $A = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$, => transpose $A' = \begin{pmatrix} 3 & 5 \\ 1 & 4 \\ 4 & 0 \\ 2 & 7 \end{pmatrix}$

Exercise 1

1. State the order of each of the following matrices:-

(a) $\begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ 4 & 8 \\ 1 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$

2. List the order of these matrices and say if any pairs are equal.

$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

$D = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $F = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $G = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $J = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ $K = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $L = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

3. Determine the values of x and y in each of the following:-

(a) $\begin{pmatrix} 3x & -y \end{pmatrix} = \begin{pmatrix} 12 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} x+3 \\ 4-y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} x+2y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$ (d) $\begin{pmatrix} x^2 & y^2 \\ y^3 & x^3 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ -27 & 8 \end{pmatrix}$

4. Write down the transpose of each matrix in question 1 and state the order of each transpose.

5. For the matrix $A = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$, show that $(A')' = A$

6. $P = \begin{pmatrix} x & 9 \\ -3 & y \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & -3 \\ 9 & -4 \end{pmatrix}$. Find x and y, given that $P' = Q$

Addition of matrices -: If two matrices A and B are of the same order, they can be added to make a new matrix A + B, formed by adding each element of A to the corresponding element of B.

In general If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ => $A + B = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$

Exercise 2

1. Find the sum of the following matrices:-

(a) $\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}4\\5\end{pmatrix}$ (b) $\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}-1\\3\end{pmatrix}$ (c) $\begin{pmatrix}2a\\b\end{pmatrix}+\begin{pmatrix}7a\\-3b\end{pmatrix}$

(d) $\begin{pmatrix}2u\\-3v\end{pmatrix}+\begin{pmatrix}-2u\\3v\end{pmatrix}$ (e) $(2 \quad 5)+(1 \quad 4)$ (f) $(2 \quad -3)+(-5 \quad 8)$

(g) $\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}2&3\\4&5\end{pmatrix}$ (h) $\begin{pmatrix}2&3&1\\5&1&0\end{pmatrix}+\begin{pmatrix}1&-2&3\\-7&1&-4\end{pmatrix}$

2. $A=\begin{pmatrix}3&2\\1&-1\end{pmatrix}$, $B=\begin{pmatrix}1&4\\5&2\end{pmatrix}$ and $C=\begin{pmatrix}-2&3\\1&-4\end{pmatrix}$

Find the matrices:-

(a) A + B (b) B + C (c) (A + B) + C (d) A + (B + C)

Is it true that (A + B) + C = A + (B + C) ?

3. Given that $A=\begin{pmatrix}3&-4\\-5&1\end{pmatrix}$ and $B=\begin{pmatrix}-3&4\\5&-1\end{pmatrix}$, find the matrices:-

(a) A + B (b) B + A

Comment on your results.

4. For the matrices $A=\begin{pmatrix}3&1&4&2\\5&4&0&7\end{pmatrix}$ and $B=\begin{pmatrix}2&-1&3&-2\\3&8&1&0\end{pmatrix}$

prove that $(A+B)' = A' + B'$?

Subtraction of matrices -: If two matrices A and B are of the same order, they can be subtracted to form a matrix A – B, formed by subtracting each element of B from the corresponding element of A.

In general If $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $B=\begin{pmatrix}p&q\\r&s\end{pmatrix}$ => $A-B=\begin{pmatrix}a-p&b-q\\c-r&d-s\end{pmatrix}$

Exercise 3

1. Subtract the following matrices.

(a) $\begin{pmatrix}3\\1\end{pmatrix}-\begin{pmatrix}4\\5\end{pmatrix}$ (b) $\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}-1\\3\end{pmatrix}$ (c) $\begin{pmatrix}2a\\b\end{pmatrix}-\begin{pmatrix}7a\\-3b\end{pmatrix}$

(d) $\begin{pmatrix}2u\\-3v\end{pmatrix}-\begin{pmatrix}-2u\\3v\end{pmatrix}$ (e) $\begin{pmatrix}2 & 5\end{pmatrix}-\begin{pmatrix}1 & 4\end{pmatrix}$ (f) $\begin{pmatrix}2 & -3\end{pmatrix}-\begin{pmatrix}-5 & 8\end{pmatrix}$

(g) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}$ (h) $\begin{pmatrix}4 & 7\\2 & 3\end{pmatrix}-\begin{pmatrix}8 & 0\\3 & 9\end{pmatrix}$ (i) $\begin{pmatrix}2 & 3 & 1\\5 & 1 & 0\end{pmatrix}-\begin{pmatrix}1 & -2 & 3\\-7 & 1 & -4\end{pmatrix}$

2. $A=\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}, B=\begin{pmatrix}-2 & 3\\0 & 1\end{pmatrix}$ and $C=\begin{pmatrix}5 & 2\\-1 & 0\end{pmatrix}$. Find in simplest form :-

(a) $A+B$ (b) $A+C$ (c) $A+B+C$
(d) $A-B$ (e) $C-B$ (f) $C-A$
(g) $(A+C)+(A+B)$ (h) $(A+C)-(A+B)$

3. Solve each of the following equations for the 2 x 2 matrix X:-

(a) $X+\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}0 & 2\\2 & 1\end{pmatrix}$ (b) $X+\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}=\begin{pmatrix}4 & -1\\3 & 2\end{pmatrix}$

4. If $A=\begin{pmatrix}1 & 2\\3 & 4\\5 & 6\end{pmatrix}, B=\begin{pmatrix}3 & -2\\1 & 5\\4 & 6\end{pmatrix}$ and $C=\begin{pmatrix}4 & 2\\1 & 0\\-3 & 5\end{pmatrix}$, simplify:-

(a) $A+B$ (b) $B-C$ (c) $(A+B)-C$ (d) $A+(B-C)$

Multiplication of a matrix by a real number –: If k is a real number and A is a matrix, then kA is a new matrix obtained by multiplying each element of A by k.

In general If $A=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ then $kA=\begin{pmatrix}ka & kb\\kc & kd\end{pmatrix}$

Example If $A=\begin{pmatrix}2 & 1 & 3\\-1 & 0 & 4\end{pmatrix}$ and $B=\begin{pmatrix}2 & -1 & 0\\3 & 4 & 5\end{pmatrix}$, show that :-

=> $3A-2B=\begin{pmatrix}2 & 5 & 9\\-9 & -8 & 2\end{pmatrix}$

$$3A-2B=3\begin{pmatrix}2 & 1 & 3\\-1 & 0 & 4\end{pmatrix}-2\begin{pmatrix}2 & -1 & 0\\3 & 4 & 5\end{pmatrix}=\begin{pmatrix}6 & 3 & 9\\-3 & 0 & 12\end{pmatrix}-\begin{pmatrix}4 & -2 & 0\\6 & 8 & 10\end{pmatrix}$$

$$=\begin{pmatrix}2 & 5 & 9\\-9 & -8 & 2\end{pmatrix}$$

Exercise 4

1. If $A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, find (a) 2A (b) 3A (c) –5A (d) –A

2. If $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{pmatrix}$, find in their simplest form:-

   (a) A – B (b) 2(A + B) (c) 2A

   (d) 2B (e) 2A + 2B (f) 6A

   (g) 3(2A) (h) 8B (i) 4(2B)

   This question demonstrates the following properties:_

   (i) $k(A \pm B) = kA \pm kB$ and (ii) $k(tA) = (kt)A$

   where k and t are real numbers.

3. If $A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$, simplify:-

   (a) 3A + 2B (b) 4A – 3B (c) 5A – 4B (d) 2(A – 5B)

4. Solve each of the following equations for the matrix X :-

   (a) $3X = \begin{pmatrix} 6 & -3 \\ 12 & 9 \end{pmatrix}$ (b) $2X + \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 2 & 8 \end{pmatrix}$

   (c) $4X - \begin{pmatrix} 3 & 1 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 0 & 13 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} - 3X = \begin{pmatrix} -5 & 10 \\ 8 & 9 \end{pmatrix}$

5. Find the matrix X in each of the following:-

   (a) $2\begin{pmatrix} 1 & -1 & 3 \\ 2 & -7 & 5 \end{pmatrix} + X = 3\begin{pmatrix} 1 & 2 & -4 \\ 3 & -5 & 1 \end{pmatrix}$ (b) $5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 3X = 4\begin{pmatrix} -4 & 7 \\ 3 & 8 \end{pmatrix}$

6. Given that $2\begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} 7 & -2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$, find p, q, r and s.

Multiplication of matrices :- Consider the two linear expressions $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

We can write these in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

We can think of this as a product of a 2 x 2 matrix and a 2 x 1 matrix.

To obtain the linear expressions from this product, proceed as follows :-

multiply each element of a row in the first matrix by the corresponding element in the column of the second matrix and then add the products to give a (2 x 1) matrix.

Examples.

1. $$\begin{pmatrix}3 & 2 & 1\\5 & -3 & 7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3x+2y+z\\5x-3y+7z\end{pmatrix}$$ this is a 2 x 1 matrix.

Second in row x Second in column

2. $$\begin{pmatrix}4 & 3 & 2\end{pmatrix}\begin{pmatrix}3\\1\\-5\end{pmatrix}=(4\times 3+3\times 1+2\times(-5))=(12+3-10)=(5)$$

Third in row x Third in column

First in row x First in column

Note:- in 1. The product is of a (2 x $\underline{3}$) matrix and a ($\underline{3}$ x 1) matrix giving a (2 x 1) matrix.

Note:- in 2. The product is of a (1 x $\underline{3}$) matrix and a ($\underline{3}$ x 1) matrix giving a (1 x 1) matrix.

In general The product of an ***(m* x *p)*** matrix and a ***(p* x 1)** matrix produces an ***(m* x 1)** matrix.

The product of an ***(m* x *p)*** matrix and an ***(p* x *n)*** matrix can be looked upon as an extension of the product of an ***(m* x *p)*** matrix and a ***(p* x 1)** matrix .

In the examples above, the number of columns in the first matrix is the same as the number of rows in the second matrix.

The product of of an ***(m* x *p)*** and an ***(p* x *n)*** should therefore produce an ***(m* x *n)*** matrix.

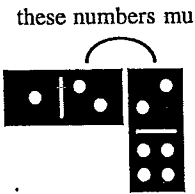
Matrices must be CONFORMABLE for Multiplication.

e.g. $$\begin{pmatrix}3 & -2 & 1\\5 & 0 & 6\end{pmatrix}\begin{pmatrix}-1 & 5\\8 & -3\\1 & 4\end{pmatrix}$$ these matrices **can** be multiplied

Order (2 x $\underline{3}$) x ($\underline{3}$ x 2) = (2 x 2) matrix.

↖ ↗
same

Think of dominoes.



The above example would then give :-

$$\begin{pmatrix}(3\times(-1)+(-2)\times 8+1\times 1) & (3\times 5+(-2)\times(-3)+1\times 4)\\(5\times(-1)+0\times 8+6\times 1) & (5\times 5+0\times(-3)+6\times 4)\end{pmatrix}=\begin{pmatrix}-18 & 25\\1 & 49\end{pmatrix}$$

Exercise 5

1. Find the following matrix products by first considering the order of the product.

(a) $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ (f) $\begin{pmatrix} 8 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ (i) $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(j) $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ (k) $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ (l) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

(m) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ (n) $\begin{pmatrix} 5 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ (o) $\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

(p) $\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ (q) $\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ (r) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

(s) $\begin{pmatrix} \cos a & \sin a \end{pmatrix}\begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}$

2. By forming a system of simultaneous equations, find x and y.

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$ (b) $\begin{pmatrix} x & 0 \\ 1 & y \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$ (d) $\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

3. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}, C = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

(a) find (i) AB (ii) BA and comment on the result.

(b) find (i) A(BC) (ii) (AB)C and comment on the result.

(c) Show that $(AB)' = B'A'$.

4. Given that $A = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}$, find A^2 and A^3. (Note :- $A^2 \neq \begin{pmatrix} 9 & 1 \\ 25 & 4 \end{pmatrix}$)

5. If $P = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, find PQ and QP.

6. Find the following products.

(a) $\begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (b) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix}$ (c) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(d) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ (e) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (f) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

The 2x2 matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is called the **unit matrix** of order 2 and is denoted by **I**.

It behaves like <u>unity</u> in the real number system. If A is a 2x2 matrix, then IA = AI = A.

7. If $\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, find a, b, c and d.

8. If $\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}2 & 1\\ 4 & 3\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, find p, q, r and s.

9. If $A = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, find p and q such that $A^2 = pA + qI$.

10. If $A = \begin{pmatrix}3 & -1\\ 2 & -5\end{pmatrix}$, find p and q such that $A^2 = pA + qI$.

11. If $A = \begin{pmatrix}1 & 1\\ 3 & 2\\ 1 & 2\end{pmatrix}$ and $B = \begin{pmatrix}2 & 1 & 3\\ 3 & -1 & 2\end{pmatrix}$, find AB and BA.

12. If $A = \begin{pmatrix}1 & 0 & 2\\ 1 & -1 & 1\\ 2 & 1 & 3\end{pmatrix}$ and $B = \begin{pmatrix}2 & -1 & 3\\ 0 & 1 & -1\\ 1 & 1 & 1\end{pmatrix}$, find AB and BA

13. Calculate M^2 and M^3 when $\theta = 60°$, given that $M = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

14. A matrix B is such that $B^2 = 6B - 9I$, where I is the 2x2 unit matrix
Find integers p and q such that $B^3 = pB + qI$.

15. A matrix A is such that $A^2 = 3A - 4I$, where I is the 2x2 unit matrix.
Find rational numbers p and q such $A^3 = pA + qI$.

Further examples covering Exs 1, 2, 3, 4, 5 can be found in the following resources :-

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 161 Exercise 6A Omit 1(m) and (n), 5, 6, 9 and 10

The determinant of a 2x2 matrix.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of the matrix A is denoted by det A or $|A|$ and defined as:- $\boxed{\det A = ad - bc}$ (a scalar quantity)

Example If $A = \begin{pmatrix} 4 & 3 \\ -1 & -2 \end{pmatrix}$ then $|A|$ or **det A** $= 4 \times (-2) - (-1) \times 3 = -5$.

If $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $|I|$ or det $I = 1$

Singular and Non-Singular Matrices

A 2x2 matrix P, is called a <u>singular</u> matrix iff $|P| = 0$ and <u>non-singular</u> iff $|P| \neq 0$.

("iff" reads "if and only if")

The determinant of a 3x3 matrix.

If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then det A is defined as follows:-

$$\boxed{\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

In this expression for det A, the factor multiplying a given element is the determinant of the matrix obtained from A by omitting the row and column which contains the given element, together with the sign, according to the chess-board pattern shown below :-

$$\boxed{\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}}$$

Example If $A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

then **det A** $= \mathbf{3}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - \mathbf{1}\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} + \mathbf{(-2)}\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= 3 \times (-2 - 1) - 1 \times (-4 - 0) + (-2) \times (2 - 0)$

$= -9 + 4 - 4$

$= -9$

Exercise 6

1. Evaluate

(a) $\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & -1 \\ -4 & -1 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$

2. If $A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$,

find (i) AB and show that det(AB) = detAdetB.

(ii) BA and show that det(AB) = det(BA).

3. Evaluate

(a) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$ (b) $\begin{vmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{vmatrix}$ (c) $\begin{vmatrix} \ln 2 & \ln 4 \\ \ln 5 & \ln 6 \end{vmatrix}$

4. Find the determinants of the following matrices.

(a) $\begin{pmatrix} -2 & 1 & 4 \\ 3 & -2 & 5 \\ 0 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 4 \\ 2 & 6 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 0 & 1 \\ 3 & -4 & 5 \\ -7 & -3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ (e) $\begin{pmatrix} -7 & 14 & 7 \\ 2 & -8 & 6 \\ 9 & -3 & 12 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 8 & -10 \\ 2 & 4 & 15 \\ 1 & 12 & 5 \end{pmatrix}$

5. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}$

find (i) AB and show that det(AB) = detAdetB.

(ii) BA and show that det(AB) = det(BA).

6. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)$.

Further examples can be found in the following resources :-

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 438 Exercise 17E Q 8

The Inverse of a 2x2 Matrix. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

then $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore AI = IA = A.$

For this reason the 2x2 unit matrix is called the identity matrix for multiplication of 2x2 matrices.

Example Consider $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$,

$AB = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore AB = BA = I$

For this reason B is called the **multiplicative inverse** of A and is denoted by A^{-1}.
In the same way, we can say that A is the (multiplicative) inverse of B and is denoted by B^{-1}.

We generally use the word **inverse** of a matrix to refer to the multiplicative inverse since the additive inverse is generally called its negative.

In general If A and B are square matrices of the same order such that AB = BA = I, then B is the inverse of A and A is the inverse of B.
It can be shown that if these inverses exist, then they are unique and so we can talk about the inverse of A or the inverse of B.

Exercise 7

1. Show that each matrix is the inverse of the other :

(a) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$ and $\begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$

2. Study the pattern in the entries of the pairs of matrices above.
Use this pattern to *write down* the inverse of each of the following matrices and check by multiplication.

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 3 \\ 9 & 7 \end{pmatrix}$

(d) $\begin{pmatrix} 9 & -5 \\ -7 & 4 \end{pmatrix}$ (e) $\begin{pmatrix} 5 & -7 \\ 3 & -4 \end{pmatrix}$ (f) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

You should have noticed in the matrices in question 2 that :-

(i) the <u>difference in the cross product of the entries</u> is always 1. (i.e. det A = 1)

In $\begin{pmatrix} 4 & 3 \\ 9 & 7 \end{pmatrix}$, $(4 \times 7) - (9 \times 3) = 1$

Note the order of the cross products
The main diagonal (4×7) first.

(ii) the inverse can be found by interchanging the entries in the main diagonal and changing the signs in the other diagonal.

Does every 2x2 Matrix have an Inverse ?

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then its inverse, A^{-1} would appear to be $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If this inverse is correct then $A^{-1}A = AA^{-1} = I$

$$A^{-1}A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad \left[\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right]\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly

$$AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that if $ad - bc \neq 0$:-

the **inverse** of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

where $ad - bc$ is the **determinant** of the matrix A.

If detA = 0, => A does <u>not</u> have an inverse and is called a <u>singular</u> matrix.

If detA $\neq$ 0, => A has an inverse and is said to be <u>non–singular</u>.

Examples

1. Given that $A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}$, find A^{-1}, if it exists.

   $\det A = 4 \times 2 - 3 \times 2 = 2 \neq 0$, so A^{-1} exists. => $A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{2} \\ -1 & 2 \end{pmatrix}$

2. Given that $P = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$, find P^{-1}, if it exists.

   $\det P = 3 \times 2 - (-2) \times (-1) = 4 \neq 0$, so P^{-1} exists. => $P^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$

Exercise 8

1. Find the inverse of the following 2x2 matrices, if they exist :

(a) $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ (b) $B = \begin{pmatrix} 7 & 4 \\ 16 & 9 \end{pmatrix}$ (c) $C = \begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$

(d) $D = \begin{pmatrix} 5 & 7 \\ 6 & 9 \end{pmatrix}$ (e) $E = \begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix}$ (f) $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

2. Given that $P = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, find

(a) P^{-1} (b) Q^{-1} (c) $(PQ)^{-1}$

(d) $P^{-1}Q^{-1}$ (e) $Q^{-1}P^{-1}$ (f) $(QP)^{-1}$

Further examples can be found in the following resources:-

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 161 Exercise 6A Q 1(m) and (n), 5, 6

The Inverse of a 3x3 Matrix. If A is a 3x3 matrix, provided A^{-1} exists, then :-

$$\Rightarrow \quad AA^{-1} = A^{-1}A = I \text{ where } I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finding the inverse of a 3x3 matrix by elementary row operations.

The method used is called "finding the inverse" if it exists by **elementary row operations.**

For a 3x3 matrix A, we somehow change the A into the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ whilst also changing $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ into the inverse of A by using row operations similar to elimination.

We begin with $A|I$ and turn this into $I|A^{-1}$.

We start to operate with I on the right hand side and by operating on A row by row we turn A into the identity matrix I, at the same time turning I into A^{-1}.

If A has no inverse, this becomes apparent because the reduction of A produces a row of 0's.

It is recommended that the product of A and A^{-1} should be found as a check

i.e. that AA^{-1} or $A^{-1}A = I$.

(If det $A = 0$, A^{-1} does not exist.)

Examples 1. Find the inverse of the 3x3 matrix $A=\begin{pmatrix}1&1&1\\1&-1&2\\2&1&-1\end{pmatrix}$

we work on the bottom triangular half of A first

(since det A = 7 => Inverse exists).

Set out as follows:-

$$\begin{array}{c|ccc|ccc} & & \mathbf{A} & & & \mathbf{I} & \\ \text{row } 1 & 1 & 1 & 1 & 1 & 0 & 0\\ \text{row } 2 & 1 & -1 & 2 & 0 & 1 & 0\\ \text{row } 3 & 2 & 1 & -1 & 0 & 0 & 1\end{array}$$

we work on the bottom triangular half of A first

$$\text{row } 3 - 2\text{x}\text{row } 1 \quad \left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&-1&2&0&1&0\\0&-1&-3&-2&0&1\end{array}\right)$$

$$\text{row } 2 - \text{row } 1 \quad \left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-2&1&-1&1&0\\0&-1&-3&-2&0&1\end{array}\right)$$

$$2\text{x}\text{row } 3 - \text{row } 2 \quad \left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-2&1&-1&1&0\\0&0&-7&-3&-1&2\end{array}\right)$$

note :- we have created a triangle of zeros

we now work on the top triangular half of A

$$7\text{x}\text{row } 1 + \text{row } 3 \quad \left(\begin{array}{ccc|ccc}7&7&0&4&-1&2\\0&-2&1&-1&1&0\\0&0&-7&-3&-1&2\end{array}\right)$$

$$7\text{x}\text{row } 2 + \text{row } 3 \quad \left(\begin{array}{ccc|ccc}7&7&0&4&-1&2\\0&-14&0&-10&6&2\\0&0&-7&-3&-1&2\end{array}\right)$$

$$2\text{x}\text{row } 1 + \text{row } 2 \quad \left(\begin{array}{ccc|ccc}14&0&0&-2&4&6\\0&-14&0&-10&6&2\\0&0&-7&-3&-1&2\end{array}\right)$$

note :- we have created a 2nd triangle of zeros

$$\begin{array}{c|ccc|ccc} & & \mathbf{I} & & & \mathbf{A^{-1}} & \\ \frac{1}{14}\times\text{row } 1 & 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{3}{7}\\ -\frac{1}{14}\times\text{row } 2 & 0 & 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7}\\ -\frac{1}{7}\times\text{row } 3 & 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & -\frac{2}{7}\end{array}$$

The inverse of $A=\begin{pmatrix}1&1&1\\1&-1&2\\2&1&-1\end{pmatrix}$ is therefore $A^{-1}=\frac{1}{7}\begin{pmatrix}-1&2&3\\5&-3&-1\\3&1&-2\end{pmatrix}$

Check your answer using $AA^{-1}=I$ or $A^{-1}A=I$

2. Find the inverse of the 3x3 matrix $P=\begin{pmatrix}2&2&1\\2&4&1\\3&2&0\end{pmatrix}$

(since det $P=-6$ => Inverse exists)

$$\begin{array}{rc} & \begin{array}{ccc|ccc} & \mathbf{A} & & & \mathbf{I} & \end{array} \\ \begin{array}{r}\text{row 1}\\ \text{row 2}\\ \text{row 3}\end{array} & \left(\begin{array}{ccc|ccc} 2&2&1&1&0&0\\ 2&4&1&0&1&0\\ 3&2&0&0&0&1\end{array}\right)\end{array}$$

$$2\text{x}\,\text{row }3-3\text{x}\,\text{row }1\quad \left(\begin{array}{ccc|ccc} 2&2&1&1&0&0\\ 2&4&1&0&1&0\\ 0&-2&-3&-3&0&2\end{array}\right)$$

$$\text{row }1-\text{row }2\quad \left(\begin{array}{ccc|ccc} 2&2&1&1&0&0\\ 0&-2&0&1&-1&0\\ 0&-2&-3&-3&0&2\end{array}\right)$$

$$\text{row }3-\text{row }2\quad \left(\begin{array}{ccc|ccc} 2&2&1&1&0&0\\ 0&-2&0&1&-1&0\\ 0&0&-3&-4&1&2\end{array}\right)$$

$$3\text{ x row }1+\text{row }3\quad \left(\begin{array}{ccc|ccc} 6&6&0&-1&1&2\\ 0&-2&0&1&-1&0\\ 0&0&-3&-4&1&2\end{array}\right)$$

$$\text{row }1+3\text{ x row }2\quad \left(\begin{array}{ccc|ccc} 6&0&0&2&-2&2\\ 0&-2&0&1&-1&0\\ 0&0&-3&-4&1&2\end{array}\right)$$

$$\begin{array}{rc} & \begin{array}{ccc|ccc} & \mathbf{I} & & & \mathbf{A^{-1}} & \end{array} \\ \begin{array}{r}\frac{1}{6}\times\text{row 1}\\[6pt] -\frac{1}{2}\times\text{row 2}\\[6pt] -\frac{1}{3}\times\text{row 3}\end{array} & \left(\begin{array}{ccc|ccc} 1&0&0&\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\[6pt] 0&1&0&-\frac{1}{2}&\frac{1}{2}&0\\[6pt] 0&0&1&\frac{4}{3}&-\frac{1}{3}&-\frac{2}{3}\end{array}\right)\end{array}$$

The inverse $P=\begin{pmatrix}2&2&1\\2&4&1\\3&2&0\end{pmatrix}$ is therefore $P^{-1}=\begin{pmatrix}\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\ -\frac{1}{2}&\frac{1}{2}&0\\ \frac{4}{3}&-\frac{1}{3}&-\frac{2}{3}\end{pmatrix}$ or $\frac{1}{6}\begin{pmatrix}2&-2&2\\-3&3&0\\8&-2&-4\end{pmatrix}$

Check your answer using $PP^{-1}=I$ or $P^{-1}P=I$

Exercise 9

Find the inverses of the following 3×3 matrices, if they exist :–

1. $\begin{pmatrix} 3 & 4 & 5 \\ 4 & 3 & 11 \\ 1 & 0 & 3 \end{pmatrix}$ 2. $\begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix}$ 3. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 4 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 5 & 1 \end{pmatrix}$ 5. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ 6. $\begin{pmatrix} 1 & 8 & 5 \\ 2 & 10 & 7 \\ 9 & 7 & 3 \end{pmatrix}$

Further examples can be found in the following resources:-

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 437 Exercise 17E Q 9

Using The Inverse of a 2x2 Matrix to Solve a System of Linear Equations

Consider the system of equations : $3x - y = 5$
$2x + y = 15$

Since $\begin{pmatrix} 3x - y \\ 2x + y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, the set of equations can be written as $\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$

which is in the form $AX = B$ where $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$

If the inverse A^{-1} of A exists => $A^{-1}AX = A^{-1}B$

i.e. $IX = A^{-1}B$ or simply $X = A^{-1}B$

In the above example, $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$

i.e. $\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 15 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 20 \\ 35 \end{pmatrix}$ => $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, and so $x = 4$ and $y = 7$

Exercise 10

Solve the following systems of equations :

1. $x - y = 5$, $x + y = 11$ 2. $3x + y = 7$, $3x + 2y = 5$ 3. $2x + y = 5$, $2x + 3y = -1$

4. $3x - 4y = 18$, $5x + y = 7$ 5. $2x + 3y = 5$, $4x - 5y = 21$ 6. $5x - 3y = 9$, $7x - 6y = 9$

Using the Inverse of a 3x3 Matrix to Solve a System of Linear Equations.

Consider the system of equations:-

$$\begin{aligned} x + y + 2z &= 3 \\ 2x - y - z &= 2 \\ 3x - 2y + 2z &= -2 \end{aligned}$$

Since $\begin{pmatrix} x+y+2z \\ 2x-y-z \\ 3x-2y+2z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we can be re-write our equations as :-

$$\boxed{\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}}$$

which is in the form AX = B where $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

If the inverse A^{-1} of A exists, $A^{-1}AX = A^{-1}B$ (i.e. $IX = A^{-1}B$)

Using the elementary row operation method shown on Pages 38 and 39,

$$A^{-1} = \frac{1}{13}\begin{pmatrix} 4 & 6 & -1 \\ 7 & 4 & -5 \\ 1 & -5 & 3 \end{pmatrix}$$

i.e. $$\frac{1}{13}\begin{pmatrix} 4 & 6 & -1 \\ 7 & 4 & -5 \\ 1 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 4 & 6 & -1 \\ 7 & 4 & -5 \\ 1 & -5 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 26 \\ 39 \\ -13 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

and so $x = 2, y = 3$ and $z = -1$

Exercise 11 Solve the following systems of equations :

1. $\begin{aligned} x - 2y + z &= 6 \\ 3x + y - 2z &= 4 \\ 7x - 6y - z &= 10 \end{aligned}$

2. $\begin{aligned} 5x - y + 2z &= 25 \\ 3x + 2y - 3z &= 16 \\ 2x - y + z &= 9 \end{aligned}$

3. $\begin{aligned} x + y + z &= 2 \\ 3x - y + 2z &= 4 \\ 2x + 3y + z &= 7 \end{aligned}$

4. $\begin{aligned} 2x + 4y + 5z &= -3 \\ 4x - y - 7z &= 6 \\ 6x + 3y - z &= 3 \end{aligned}$

Further examples can be found in the following resources:-

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 161 Exercise 6A Q 4(select), 9. Page 437 Exercise 17E Q 11, 12

Using 2×2 Matrices to Represent Geometrical Transformations in the (x, y) Plane.

2×2 matrices can be associated with transformations of all points in a Cartesian plane.
Transformations are defined as reflections in the x-axis, y-axis and in the line $y = x$, rotations of 90° and 180°, and dilatation (enlargement or reduction), or compositions of these.

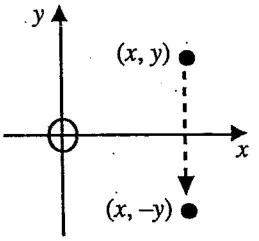
Examples

1. Reflection of the point (x, y) in the x–axis.

Under this reflection (x, y) maps on to (x', y') where (x', y') is $(x, -y)$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1x + 0y \\ 0x - 1y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and so $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with reflection in the x-axis.

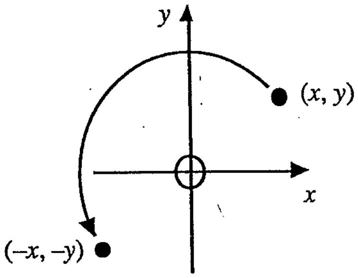


2. Rotation of π radians (a half-turn) about the origin.

Under this rotation (x, y) maps on to (x', y'), where (x', y') is $(-x, -y)$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} -1x + 0y \\ 0x - 1y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and so $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with a half-turn rotation about the origin.

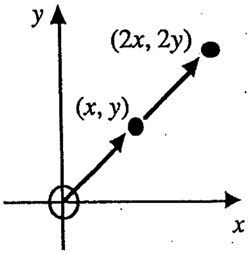


3. A dilatation where the scale factor is 2 and the centre of dilatation is the origin.

Under this dilatation (x, y) maps on to (x', y'), where (x', y') is $(2x, 2y)$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 2x + 0y \\ 0x + 2y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and so $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is the matrix associated with a dilatation, centre the origin and with a scale factor of 2.



Exercise 12

1. Find the matrices associated with the following transformations.

   (a) Reflection in the y – axis.

   (b) Reflection in the line $y = x$.

   (c) Reflection in the line $y = -x$.

   (d) A rotation of $\frac{\pi}{2}$ radians clockwise.

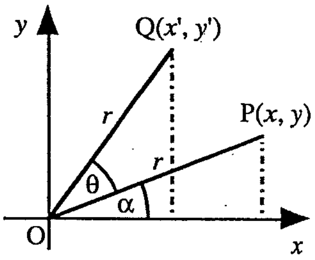
   (e) A rotation of $\frac{\pi}{2}$ radians anti–clockwise.

   (f) A dilatation, about O, where the scale factor is k.

2. Prove that the matrix associated with a general rotation of θ radians about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

   The diagram opposite may be helpful.

   OP, which makes an angle of α radians with the x–axis, is rotated through θ radians to OQ.



   Hence, show that the matrix associated with a rotation of:-

   (i) π radians is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

   (ii) $\frac{\pi}{2}$ radians anti–clockwise is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

   (iii) $\frac{\pi}{2}$ radians clockwise is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Further examples can be found in the following resources :-

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 168 Exercise 6B Select

Answers

Exercise 1

1. (a) 2 x 4 (b) 3 x 2 (c) 4 x 2

2. A = C (1 x 3), F = G (2 x 1), H = L (2 x 2)

3. (a) $x = 4, y = -3$ (b) $x = 4, y = -1$ (c) $x = 5, y = 2$ (d) $x = 2, y = -3$

4. (a) $\begin{pmatrix} 3 & 5 \\ 1 & 4 \\ 4 & 0 \\ 2 & 7 \end{pmatrix}$, 4 x 2 (b) $\begin{pmatrix} 2 & 4 & 1 \\ -1 & 8 & -2 \end{pmatrix}$, 2 x 3 (c) $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$, 2 x 4

5. Proof 6. $x = 5, y = -4$

Exercise 2

1. (a) $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ (c) $\begin{pmatrix} 9a \\ -2b \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 3 & 9 \end{pmatrix}$ (f) $\begin{pmatrix} -3 & 5 \end{pmatrix}$ (g) $\begin{pmatrix} 3 & 3 \\ 4 & 6 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 1 & 4 \\ -2 & 2 & -4 \end{pmatrix}$

2. (a) $\begin{pmatrix} 4 & 6 \\ 6 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 7 \\ 6 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 9 \\ 7 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 9 \\ 7 & -3 \end{pmatrix}$,true

3. A = –B 4. Proof

Exercise 3

1. (a) $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} -5a \\ 4b \end{pmatrix}$ (d) $\begin{pmatrix} 4u \\ -6v \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 7 & -11 \end{pmatrix}$ (g) $\begin{pmatrix} -1 & -3 \\ -4 & -4 \end{pmatrix}$ (h) $\begin{pmatrix} -4 & 7 \\ -1 & -6 \end{pmatrix}$

   (i) $\begin{pmatrix} 1 & 5 & -2 \\ 12 & 0 & 4 \end{pmatrix}$

2. (a) $\begin{pmatrix} -1 & 5 \\ 3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 4 \\ 2 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -1 \\ 3 & 3 \end{pmatrix}$

   (e) $\begin{pmatrix} 7 & -1 \\ -1 & -1 \end{pmatrix}$ (f) $\begin{pmatrix} 4 & 0 \\ -4 & -4 \end{pmatrix}$ (g) $\begin{pmatrix} 5 & 9 \\ 5 & 9 \end{pmatrix}$ (h) $\begin{pmatrix} 7 & -1 \\ -1 & -1 \end{pmatrix}$

3. (a) $\begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -4 \\ -1 & -3 \end{pmatrix}$

4. (a) $\begin{pmatrix} 4 & 0 \\ 4 & 9 \\ 9 & 12 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -4 \\ 0 & 5 \\ 7 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -2 \\ 3 & 9 \\ 12 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & -2 \\ 3 & 9 \\ 12 & 7 \end{pmatrix}$

Exercise 4

1. (a) $\begin{pmatrix}6 & 8\\ -2 & -4\end{pmatrix}$ (b) $\begin{pmatrix}9 & 12\\ -3 & -6\end{pmatrix}$ (c) $\begin{pmatrix}-15 & -20\\ 5 & 10\end{pmatrix}$ (d) $\begin{pmatrix}-3 & -4\\ 1 & 2\end{pmatrix}$

2. (a) $\begin{pmatrix}1 & 5 & -2\\ -2 & -5 & 3\end{pmatrix}$ (b) $\begin{pmatrix}10 & 6 & 8\\ 12 & 10 & 6\end{pmatrix}$ (c) $\begin{pmatrix}6 & 8 & 2\\ 4 & 0 & 6\end{pmatrix}$

   (d) $\begin{pmatrix}4 & -2 & 6\\ 8 & 10 & 0\end{pmatrix}$ (e) $\begin{pmatrix}10 & 6 & 8\\ 12 & 10 & 6\end{pmatrix}$ (f) $\begin{pmatrix}18 & 24 & 6\\ 12 & 0 & 18\end{pmatrix}$

   (g) $\begin{pmatrix}18 & 24 & 6\\ 12 & 0 & 18\end{pmatrix}$ (h) $\begin{pmatrix}16 & -8 & 24\\ 32 & 40 & 0\end{pmatrix}$ (i) $\begin{pmatrix}16 & -8 & 24\\ 32 & 40 & 0\end{pmatrix}$

3. (a) $\begin{pmatrix}-2 & -7\\ 18 & -1\end{pmatrix}$ (b) $\begin{pmatrix}20 & -15\\ 7 & 10\end{pmatrix}$ (c) $\begin{pmatrix}26 & -19\\ 8 & 13\end{pmatrix}$ (d) $\begin{pmatrix}44 & -16\\ -22 & 22\end{pmatrix}$

4. (a) $\begin{pmatrix}2 & -1\\ 4 & 3\end{pmatrix}$ (b) $\begin{pmatrix}3 & 2\\ -1 & 3\end{pmatrix}$ (c) $\begin{pmatrix}2 & 1\\ 1 & 5\end{pmatrix}$ (d) $\begin{pmatrix}4 & -3\\ -4 & -2\end{pmatrix}$

5. (a) $\begin{pmatrix}1 & 8 & -18\\ 5 & -1 & -7\end{pmatrix}$ (b) $\begin{pmatrix}7 & -6\\ 1 & -4\end{pmatrix}$

6. $p = -1, q = 4, r = 3, s = -2$

Exercise 5

1. (a) (11) (b) (26) (c) (9) (d) (13)

   (e) (15) (f) $(8x - 5y - z)$ (g) $\begin{pmatrix}9\\ 8\end{pmatrix}$ (h) $\begin{pmatrix}10\\ 4\end{pmatrix}$

   (i) $\begin{pmatrix}1\\ 8\end{pmatrix}$ (j) $\begin{pmatrix}7\\ 5\end{pmatrix}$ (k) $\begin{pmatrix}-7\\ 7\end{pmatrix}$ (l) $\begin{pmatrix}4\\ 6\end{pmatrix}$

   (m) $\begin{pmatrix}1\\ 22\end{pmatrix}$ (n) not possible (o) (26) (p) $\begin{pmatrix}11\\ 13\\ 7\end{pmatrix}$

   (q) not possible (r) $\begin{pmatrix}9\\ 8\\ 7\end{pmatrix}$ (s) $\begin{pmatrix}1\\ 0\end{pmatrix}$

2. (a) $x = 4,\ y = -4$ (b) $x = 3, y = -1$ (c) $x = 8, y = 5$ (d) $x = 2, y = -1$

3. (a) (i) $\begin{pmatrix}8 & 5\\ 14 & 15\end{pmatrix}$ (ii) $\begin{pmatrix}19 & 13\\ 2 & 4\end{pmatrix}$ $AB \neq BA$

   (b) (i) $\begin{pmatrix}17\\ 11\end{pmatrix}$ (ii) $\begin{pmatrix}17\\ 11\end{pmatrix}$ $A(BC) = (AB)C$

   (c) Proof

4. $\begin{pmatrix}4 & -5\\ 25 & -1\end{pmatrix}, \begin{pmatrix}-13 & -14\\ 70 & -27\end{pmatrix}$

5. $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}-4 & 8\\ -2 & 4\end{pmatrix}$

6. (a) $\begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix}$ (b) $\begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix}$ (c) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ (d) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

(e) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (f) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

7. $a = 1, b = -1, c = -2, d = 3$

8. $p = \frac{3}{2}, q = -\frac{1}{2}, r = -2, s = 1$

9. $p = 5, q = 2$

10. $\mathrm{p} = -2, \mathrm{q} = 13$

11. $\begin{pmatrix}5 & 0 & 5\\ 12 & 1 & 13\\ 8 & -1 & 7\end{pmatrix}, \begin{pmatrix}8 & 10\\ 2 & 5\end{pmatrix}$

12. $\begin{pmatrix}4 & 1 & 5\\ 3 & -1 & 5\\ 7 & 2 & 8\end{pmatrix}, \begin{pmatrix}7 & 4 & 12\\ -1 & -2 & -2\\ 4 & 0 & 6\end{pmatrix}$

13. $\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

14. $p = 27, q = -54$

15. $p = 5, q = -12$

Exercise 6

1. (a) 13 (b) – 5 (c) – 6 (d) 0

2. Proofs

3. (a) 1 (b) $\cos 4\theta$ (c) $\ln 2 . \ln\frac{6}{25}$ (from ln2ln6 – ln5ln4)

4. (a) 25 (b) 14 (c) 51 (d) 0

(e) 1428 (f) – 320

5. Proofs

6. Proof

Exercise 7

1. Proofs

2. (a) $\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$ (b) $\begin{pmatrix}5 & -3\\ -3 & 2\end{pmatrix}$ (c) $\begin{pmatrix}7 & -3\\ -9 & 4\end{pmatrix}$ (d) $\begin{pmatrix}4 & 5\\ 7 & 9\end{pmatrix}$

(e) $\begin{pmatrix}-4 & 7\\ -3 & 5\end{pmatrix}$ (f) $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

Exercise 8

1. (a) $\begin{pmatrix}\frac{3}{2} & -\frac{1}{2}\\ -2 & 1\end{pmatrix}$ (b) $\begin{pmatrix}-9 & 4\\ 16 & -7\end{pmatrix}$ (c) does not exist (d) $\begin{pmatrix}3 & -\frac{7}{3}\\ -2 & \frac{5}{3}\end{pmatrix}$

(e) $\begin{pmatrix}\frac{1}{2} & 2\\ \frac{1}{2} & 1\end{pmatrix}$ (f) $\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}$

2. (a) $\begin{pmatrix}\frac{1}{2} & -\frac{3}{2}\\ 0 & 1\end{pmatrix}$ (b) $\begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$ (c) $\begin{pmatrix}\frac{3}{2} & -\frac{19}{2}\\ -\frac{1}{2} & \frac{7}{2}\end{pmatrix}$ (d) $\begin{pmatrix}3 & -\frac{11}{2}\\ -1 & 2\end{pmatrix}$

(e) $\begin{pmatrix}\frac{3}{2} & -\frac{19}{2}\\ -\frac{1}{2} & \frac{7}{2}\end{pmatrix}$ (f) $\begin{pmatrix}3 & -\frac{11}{2}\\ -1 & 2\end{pmatrix}$

Exercise 9

1. $\begin{pmatrix}\frac{9}{8} & -\frac{3}{2} & \frac{29}{8}\\ -\frac{1}{8} & \frac{1}{2} & -\frac{13}{8}\\ -\frac{3}{8} & \frac{1}{2} & -\frac{7}{8}\end{pmatrix}$ 2. $\begin{pmatrix}11 & -12 & -7\\ -8 & 9 & 5\\ 7 & -8 & -4\end{pmatrix}$ 3. $\begin{pmatrix}0 & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & -1\\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$

4. $\begin{pmatrix}\frac{1}{2} & -\frac{1}{6} & -\frac{1}{6}\\ -\frac{1}{6} & -\frac{1}{18} & \frac{5}{18}\\ -\frac{2}{3} & \frac{7}{9} & \frac{1}{9}\end{pmatrix}$ 5. $\begin{pmatrix}-\frac{1}{12} & -\frac{1}{3} & \frac{7}{12}\\ -\frac{1}{12} & \frac{2}{3} & -\frac{5}{12}\\ \frac{5}{12} & -\frac{1}{3} & \frac{1}{12}\end{pmatrix}$ 6. $\begin{pmatrix}-\frac{1}{3} & \frac{11}{57} & \frac{2}{19}\\ 1 & -\frac{14}{19} & \frac{1}{19}\\ -\frac{4}{3} & \frac{65}{57} & -\frac{2}{19}\end{pmatrix}$

Exercise 10

1. $x = 8, y = 3$
2. $x = 3, y = -2$
3. $x = 4, y = -3$
4. $x = 2, y = -3$
5. $x = 4, y = -1$
6. $x = 3, y = 2$

Exercise 11

1. $x = 5, y = 3, z = 7$
2. $x = 5, y = 2, z = 1$
3. $x = 3, y = 1, z = -2$
4. $x = \frac{7}{6}, y = -\frac{4}{3}, z = 0$

Exercise 12

1. (a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

   (d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

2. Proofs