Higher

Mathematics

Polynomials and Quadratics

Contents

CfE Edition

Polynomials and Quadratics

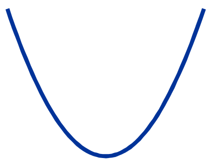
1 Quadratics

A **quadratic** has the form $ax^2 + bx + c$ where a, b, and c are any real numbers, provided $a \neq 0$.

You should already be familiar with the following.

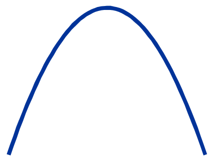
The graph of a quadratic is called a **parabola**. There are two possible shapes:

concave up (if $a > 0$)



This has a minimum turning point

concave down (if $a < 0$)



This has a maximum turning point

To find the roots (i.e. solutions) of the quadratic equation $ax^2 + bx + c = 0$, we can use:

- factorisation;
- completing the square (see Section 3);
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (this is *not* given in the exam).

EXAMPLES

1. Find the roots of $x^2 - 2x - 3 = 0$.

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ x + 1 = 0 \quad &\text{or} \quad x - 3 = 0 \\ x = -1 \qquad & \qquad x = 3. \end{aligned}$$

2. Solve $x^2 + 8x + 16 = 0$.

$$\begin{aligned} x^2 + 8x + 16 &= 0 \\ (x+4)(x+4) &= 0 \\ x + 4 = 0 \quad &\text{or} \quad x + 4 = 0 \\ x = -4 \qquad & \qquad x = -4. \end{aligned}$$

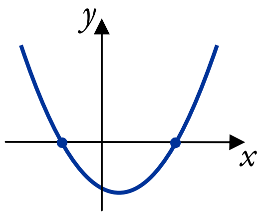
3. Find the roots of $x^2 + 4x - 1 = 0$.

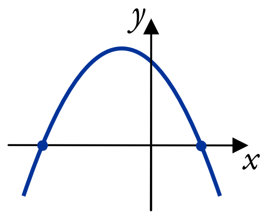
We cannot factorise $x^2 + 4x - 1$, but we can use the quadratic formula:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\ &= \frac{-4 \pm \sqrt{16 + 4}}{2} \\ &= \frac{-4 \pm \sqrt{20}}{2} \\ &= -\tfrac{4}{2} \pm \tfrac{\sqrt{4}\sqrt{5}}{2} \\ &= -2 \pm \sqrt{5}. \end{aligned}$$

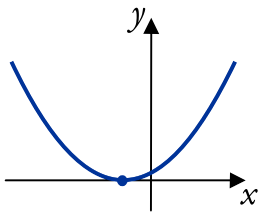
Note

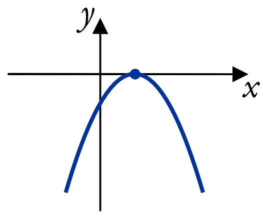
- If there are two distinct solutions, the curve intersects the x-axis twice.



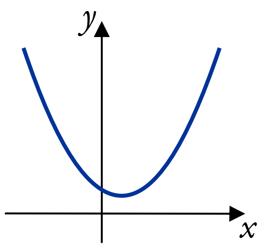


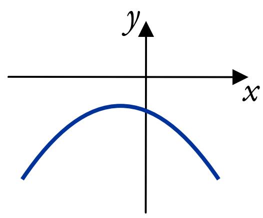
- If there is one repeated solution, the turning point lies on the x-axis.





- If $b^2 - 4ac < 0$ when using the quadratic formula, there are no points where the curve intersects the x-axis.



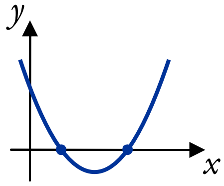


2 The Discriminant

Given $ax^2 + bx + c$, we call $b^2 - 4ac$ the **discriminant**.

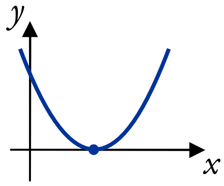
This is the part of the quadratic formula which determines the number of real roots of the equation $ax^2 + bx + c = 0$.

- If $b^2 - 4ac > 0$, the roots are real and unequal (distinct).



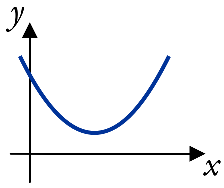
two roots

- If $b^2 - 4ac = 0$, the roots are real and equal (i.e. a repeated root).



one root

- If $b^2 - 4ac < 0$, the roots are not real; the parabola does not cross the x-axis.



no real roots

EXAMPLE

1. Find the nature of the roots of $9x^2 + 24x + 16 = 0$.

$a = 9$, $b = 24$, $c = 16$

$$\begin{aligned} b^2 - 4ac &= 24^2 - 4 \times 9 \times 16 \\ &= 576 - 576 \\ &= 0 \end{aligned}$$

Since $b^2 - 4ac = 0$, the roots are real and equal.

2. Find the values of q such that $6x^2 + 12x + q = 0$ has real roots.

Since $6x^2 + 12x + q = 0$ has real roots, $b^2 - 4ac \geq 0$:

$a = 6$, $b = 12$, $c = q$

$$\begin{aligned} b^2 - 4ac &\geq 0 \\ 12^2 - 4 \times 6 \times q &\geq 0 \\ 144 - 24q &\geq 0 \\ 144 &\geq 24q \\ 24q &\leq 144 \\ q &\leq 6. \end{aligned}$$

3. Find the range of values of k for which the equation $kx^2 + 2x - 7 = 0$ has no real roots.

For no real roots, we need $b^2 - 4ac < 0$:

$a = k$

$b = 2$

$c = -7$

$$\begin{aligned} b^2 - 4ac &< 0 \\ 2^2 - 4 \times k \times (-7) &< 0 \\ 4 + 28k &< 0 \\ 28k &< -4 \\ k &< -\frac{4}{28} \\ k &< -\tfrac{1}{7}. \end{aligned}$$

4. Show that $(2k+4)x^2 + (3k+2)x + (k-2) = 0$ has real roots for all real values of k.

$a = 2k + 4$

$b = 3k + 2$

$c = k - 2$

$$\begin{aligned} & b^2 - 4ac \\ &= (3k+2)^2 - 4(2k+4)(k-2) \\ &= 9k^2 + 12k + 4 - (2k+4)(4k-8) \\ &= 9k^2 + 12k + 4 - 8k^2 + 32 \\ &= k^2 + 12k + 36 \\ &= (k+6)^2. \end{aligned}$$

Since $b^2 - 4ac = (k+6)^2 \geq 0$, the roots are always real.

3 Completing the Square

The process of writing $y = ax^2 + bx + c$ in the form $y = a(x+p)^2 + q$ is called **completing the square**.

Once in “completed square” form we can determine the turning point of any parabola, including those with no real roots.

The axis of symmetry is $x = -p$ and the turning point is $(-p, q)$.

The process relies on the fact that $(x+p)^2 = x^2 + 2px + p^2$. For example, we can write the expression $x^2 + 4x$ using the bracket $(x+2)^2$ since when multiplied out this gives the terms we want – with an extra constant term.

This means we can rewrite the expression $x^2 + kx$ using $\left(x + \frac{k}{2}\right)^2$ since this gives us the correct x^2 and x terms, with an extra constant.

We will use this to help complete the square for $y = 3x^2 + 12x - 3$.

Step 1

Make sure the equation is in the form $y = ax^2 + bx + c$.

$$y = 3x^2 + 12x - 3.$$

Step 2

Take out the x^2-coefficient as a factor of the x^2 and x terms.

$$y = 3\left(x^2 + 4x\right) - 3.$$

Step 3

Replace the $x^2 + kx$ expression and compensate for the extra constant.

$$\begin{aligned} y &= 3\left((x+2)^2 - 4\right) - 3 \\ &= 3(x+2)^2 - 12 - 3. \end{aligned}$$

Step 4

Collect together the constant terms.

$$y = 3(x+2)^2 - 15.$$

Now that we have completed the square, we can see that the parabola with equation $y = 3x^2 + 12x - 3$ has turning point $(-2, -15)$.

EXAMPLES

1. Write $y = x^2 + 6x - 5$ in the form $y = (x+p)^2 + q$.

$$\begin{aligned} y &= x^2 + 6x - 5 \\ &= (x+3)^2 - 9 - 5 \\ &= (x+3)^2 - 14. \end{aligned}$$

Note
You can always check your answer by expanding the brackets.

2. Write $x^2 + 3x - 4$ in the form $(x+p)^2 + q$.

$$\begin{aligned} &x^2 + 3x - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{25}{4}. \end{aligned}$$

3. Write $y = x^2 + 8x - 3$ in the form $y = (x + a)^2 + b$ and then state:
   (i) the axis of symmetry, and
   (ii) the minimum turning point of the parabola with this equation.

$$\begin{aligned} y &= x^2 + 8x - 3 \\ &= (x+4)^2 - 16 - 3 \\ &= (x+4)^2 - 19. \end{aligned}$$

(i) The axis of symmetry is $x = -4$.

(ii) The minimum turning point is $(-4, -19)$.

4. A parabola has equation $y = 4x^2 - 12x + 7$.
   (a) Express the equation in the form $y = (x + a)^2 + b$.
   (b) State the turning point of the parabola and its nature.

(a)
$$\begin{aligned} y &= 4x^2 - 12x + 7 \\ &= 4\left(x^2 - 3x\right) + 7 \\ &= 4\left(\left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4}\right) + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 9 + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 2. \end{aligned}$$

(b) The turning point is $\left(\tfrac{3}{2}, -2\right)$ and is a minimum.

Remember
If the coefficient of x^2 is positive then the parabola is concave up.

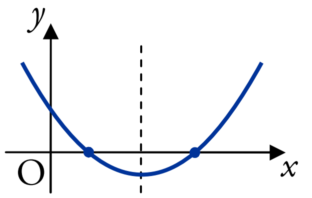
4 Sketching Parabolas

The method used to sketch the curve with equation $y = ax^2 + bx + c$ depends on how many times the curve intersects the x-axis.

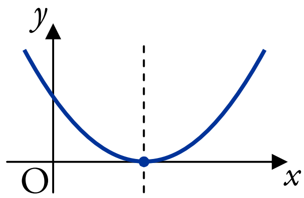
We have met curve sketching before. However, when sketching parabolas, we *do not* need to use calculus. We know there is only one turning point, and we have methods for finding it.

Parabolas with one or two roots

- Find the x-axis intercepts by factorising or using the quadratic formula.
- Find the y-axis intercept (i.e. where $x = 0$).
- The turning point is on the axis of symmetry:



The axis of symmetry is halfway between two distinct roots.



A repeated root lies on the axis of symmetry.

Parabolas with no real roots

- There are no x-axis intercepts.
- Find the y-axis intercept (i.e. where $x = 0$).
- Find the turning point by completing the square.

EXAMPLES

1. Sketch the graph of $y = x^2 - 8x + 7$.

Since $b^2 - 4ac = (-8)^2 - 4 \times 1 \times 7 > 0$, the parabola crosses the x-axis twice.

The y-axis intercept $(x = 0)$:

$$y = (0)^2 - 8(0) + 7$$
$$= 7$$

$(0, 7)$.

The x-axis intercepts $(y = 0)$:

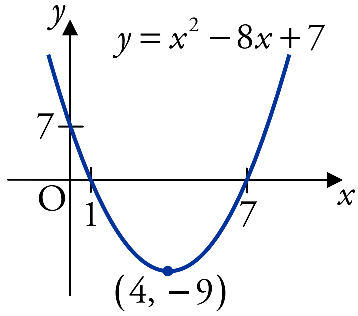
$$x^2 - 8x + 7 = 0$$
$$(x-1)(x-7) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 7 = 0$$
$$x = 1 \qquad\qquad x = 7$$
$$(1, 0) \qquad\qquad (7, 0).$$

The axis of symmetry lies halfway between $x = 1$ and $x = 7$, i.e. $x = 4$, so the x-coordinate of the turning point is 4.

We can now find the y-coordinate:

$$y = (4)^2 - 8(4) + 7$$
$$= 16 - 32 + 7$$
$$= -9.$$

So the turning point is $(4, -9)$.



2. Sketch the parabola with equation $y = -x^2 - 6x - 9$.

Since $b^2 - 4ac = (-6)^2 - 4 \times (-1) \times (-9) = 0$, there is a repeated root.

The y-axis intercept $(x = 0)$:

$$y = -(0)^2 - 6(0) - 9$$
$$= -9$$

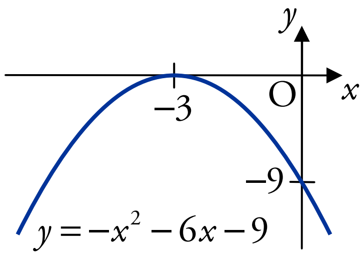
$(0, -9)$.

The x-axis intercept $(y = 0)$:

$$-x^2 - 6x - 9 = 0$$
$$-(x^2 + 6x + 9) = 0$$
$$(x+3)(x+3) = 0$$
$$x + 3 = 0$$
$$x = -3$$

$(-3, 0)$.

Since there is a repeated root, $(-3, 0)$ is the turning point.



3. Sketch the curve with equation $y = 2x^2 - 8x + 13$.

Since $b^2 - 4ac = (-8)^2 - 4 \times 2 \times 13 < 0$, there are no real roots.

The y-axis intercept $(x = 0)$:

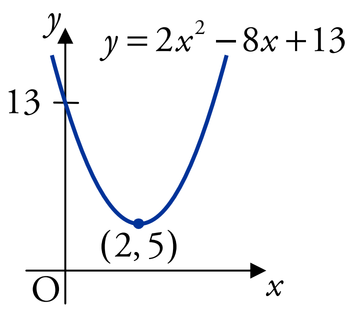
$$y = 2(0)^2 - 8(0) + 13$$
$$= 13$$

$(0, 13)$.

Complete the square:

$$y = 2x^2 - 8x + 13$$
$$= 2(x^2 - 4x) + 13$$
$$= 2(x-2)^2 - 8 + 13$$
$$= 2(x-2)^2 + 5.$$

So the turning point is $(2, 5)$.



5 Determining the Equation of a Parabola

Given the equation of a parabola, we have seen how to sketch its graph. We will now consider the opposite problem: finding an equation for a parabola based on information about its graph.

We can find the equation given:

- the roots and another point,
- the turning point and another point.

When we know the roots

If a parabola has roots $x = a$ and $x = b$ then its equation is of the form

$$y = k(x-a)(x-b)$$

where k is some constant.

If we know another point on the parabola, then we can find the value of k.

EXAMPLES

1. A parabola passes through the points $(1, 0)$, $(5, 0)$ and $(0, 3)$.
   Find the equation of the parabola.

Since the parabola cuts the x-axis where $x = 1$ and $x = 5$, the equation is of the form:

$$y = k(x-1)(x-5).$$

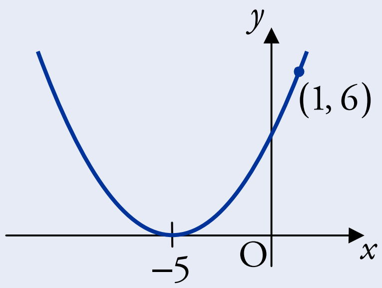
To find k, we use the point $(0, 3)$:

$$y = k(x-1)(x-5)$$
$$3 = k(0-1)(0-5)$$
$$3 = 5k$$
$$k = \tfrac{3}{5}.$$

So the equation of the parabola is:

$$\begin{aligned} y &= \tfrac{3}{5}(x-1)(x-5) \\ &= \tfrac{3}{5}(x^2 - 6x + 5) \\ &= \tfrac{3}{5}x^2 - \tfrac{18}{5}x + 3. \end{aligned}$$

2. Find the equation of the parabola shown below.



Since there is a repeated root, the equation is of the form:

$$y = k(x+5)(x+5)$$
$$= k(x+5)^2.$$

To find k, we use $(1, 6)$:

$$y = k(x+5)^2$$
$$6 = k(1+5)^2$$
$$k = \tfrac{6}{6^2} = \tfrac{1}{6}.$$

Hence $y = \frac{1}{6}(x+5)^2$.

When we know the turning point

Recall from Completing the Square that a parabola with turning point $(-p, q)$ has an equation of the form

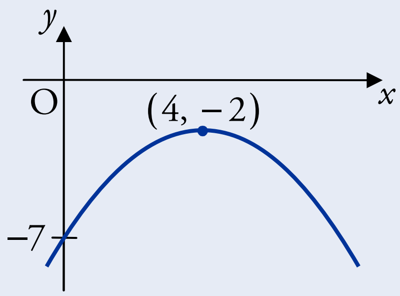
$$y = a(x+p)^2 + q$$

where a is some constant.

If we know another point on the parabola, then we can find the value of a.

EXAMPLE

3. Find the equation of the parabola shown below.



Since the turning point is $(4, -2)$, the equation is of the form:

$$y = a(x-4)^2 - 2.$$

To find a, we use $(0, -7)$:

$$y = a(x-4)^2 - 2$$
$$-7 = a(0-4)^2 - 2$$
$$16a = -5$$
$$a = -\tfrac{5}{16}.$$

Hence $y = -\frac{5}{16}(x-4)^2 - 2$.

6 Solving Quadratic Inequalities

The most efficient way of solving a quadratic inequality is by making a rough sketch of the parabola. To do this we need to know:

- the shape – concave up or concave down,
- the x-axis intercepts.

We can then solve the quadratic inequality by inspection of the sketch.

EXAMPLES

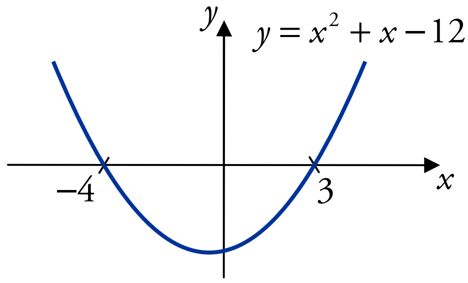
1. Solve $x^2 + x - 12 < 0$.

The parabola with equation $y = x^2 + x - 12$ is concave up.

The x-axis intercepts are given by:

$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -4 \qquad\qquad x = 3.$$

Make a sketch:



So $x^2 + x - 12 < 0$ for $-4 < x < 3$.

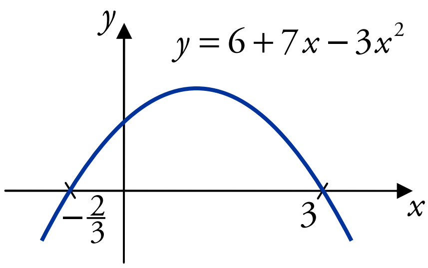
2. Find the values of x for which $6 + 7x - 3x^2 \geq 0$.

The parabola with equation $y = 6 + 7x - 3x^2$ is concave down.

The x-axis intercepts are given by:

$$6 + 7x - 3x^2 = 0$$
$$-(3x^2 - 7x - 6) = 0$$
$$(3x+2)(x-3) = 0$$
$$3x + 2 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -\tfrac{2}{3} \qquad\qquad x = 3.$$

Make a sketch:



So $6 + 7x - 3x^2 \geq 0$ for $-\tfrac{2}{3} \leq x \leq 3$.

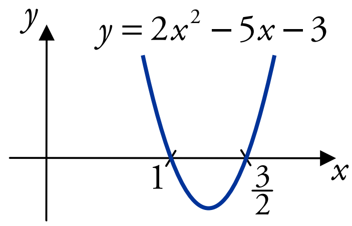
3. Solve $2x^2 - 5x - 3 > 0$.

The parabola with equation $y = 2x^2 - 5x - 3$ is concave up.

The x-axis intercepts are given by:

$$2x^2 - 5x - 3 = 0$$
$$(x-1)(2x-3) = 0$$
$$x - 1 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$x = 1 \qquad\qquad x = \tfrac{3}{2}.$$

Make a sketch:



So $2x^2 - 5x - 3 > 0$ for $x < 1$ and $x > \frac{3}{2}$.

4. Find the range of values of x for which the curve $y = \frac{1}{3}x^3 + 2x^2 - 5x + 3$ is strictly increasing.

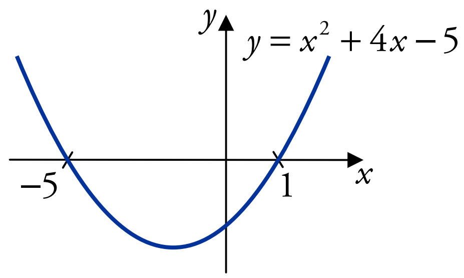
We have $\dfrac{dy}{dx} = x^2 + 4x - 5$.

Remember
Strictly increasing means $\dfrac{dy}{dx} > 0$.

The curve is strictly increasing where $x^2 + 4x - 5 > 0$.

$$x^2 + 4x - 5 = 0$$
$$(x-1)(x+5) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = 1 \qquad\qquad x = -5.$$

Make a sketch:



So the curve is strictly increasing for $x < -5$ and $x > 1$.

5. Find the values of q for which $x^2 + (q-4)x + \frac{1}{2}q = 0$ has no real roots.

For no real roots, $b^2 - 4ac < 0$:

$a = 1$

$b = q - 4$

$c = \frac{1}{2}q$

$$\begin{aligned} b^2 - 4ac &= (q-4)^2 - 4(1)\left(\tfrac{1}{2}q\right) \\ &= (q-4)(q-4) - 2q \\ &= q^2 - 8q + 16 - 2q \\ &= q^2 - 10q + 16. \end{aligned}$$

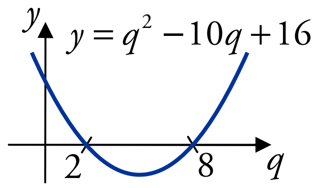
We now need to solve the inequality $q^2 - 10q + 16 < 0$.

The parabola with equation $y = q^2 - 10q + 16$ is concave up.

The x-axis intercepts are given by:

$$q^2 - 10q + 16 = 0$$
$$(q-2)(q-8) = 0$$
$$q - 2 = 0 \quad \text{or} \quad q - 8 = 0$$
$$q = 2 \qquad\qquad q = 8.$$

Make a sketch:

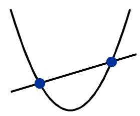


Therefore $b^2 - 4ac < 0$ for $2 < q < 8$, and so $x^2 + (q-4)x + \frac{1}{2}q = 0$ has no real roots when $2 < q < 8$.

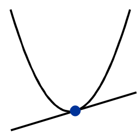
7 Intersections of Lines and Parabolas

To determine how many times a line intersects a parabola, we substitute the equation of the line into the equation of the parabola. We can then use the discriminant, or factorisation, to find the number of intersections.

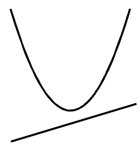
- If $b^2 - 4ac > 0$, the line and curve intersect twice.



- If $b^2 - 4ac = 0$, the line and curve intersect once (i.e. the line is a tangent to the curve).



- If $b^2 - 4ac < 0$, the line and the parabola do not intersect.



EXAMPLES

1. Show that the line $y = 5x - 2$ is a tangent to the parabola $y = 2x^2 + x$ and find the point of contact.

Substitute $y = 5x - 2$ into:

$$y = 2x^2 + x$$
$$5x - 2 = 2x^2 + x$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)(x-1) = 0.$$

Since there is a repeated root, the line is a tangent at $x = 1$.

To find the y-coordinate, substitute $x = 1$ into the equation of the line:

$$y = 5 \times 1 - 2 = 3.$$

So the point of contact is $(1, 3)$.

2. Find the equation of the tangent to $y = x^2 + 1$ that has gradient 3.

The equation of the tangent is of the form $y = mx + c$, with $m = 3$, i.e. $y = 3x + c$.

Substitute this into $y = x^2 + 1$

$$3x + c = x^2 + 1$$
$$x^2 - 3x + 1 - c = 0.$$

Since the line is a tangent:

$$b^2 - 4ac = 0$$
$$(-3)^2 - 4 \times (1 - c) = 0$$
$$9 - 4 + 4c = 0$$
$$4c = -5$$
$$c = -\tfrac{5}{4}.$$

Therefore the equation of the tangent is:

$$y = 3x - \tfrac{5}{4}$$
$$3x - y - \tfrac{5}{4} = 0.$$

Note
You could also do this question using methods from Differentiation.

8 Polynomials

Polynomials are expressions with one or more terms added together, where each term has a number (called the **coefficient**) followed by a variable (such as *x*) raised to a whole number power. For example:

$$3x^5 + x^3 + 2x^2 - 6 \quad \text{or} \quad 2x^{18} + 10.$$

The **degree** of the polynomial is the value of its highest power, for example:

$3x^5 + x^3 + 2x^2 - 6$ has degree 5 $\qquad$ $2x^{18} + 10$ has degree 18.

Note that quadratics are polynomials of degree two. Also, constants are polynomials of degree zero (e.g. 6 is a polynomial, since $6 = 6x^0$).

9 Synthetic Division

Synthetic division provides a quick way of evaluating polynomials.

For example, consider $f(x) = 2x^3 - 9x^2 + 2x + 1$. Evaluating directly, we find $f(6) = 121$. We can also evaluate this using synthetic division with detached coefficients.

Step 1

Detach the coefficients, and write them across the top row of the table.

Note that they must be in order of *decreasing* degree. If there is no term of a specific degree, then zero is its coefficient.

	2	–9	2	1

Step 2

Write the number for which you want to evaluate the polynomial (the input number) to the left.

6	2	–9	2	1

Step 3

Bring down the first coefficient.

6	2	–9	2	1
	2			

Step 4

Multiply this by the input number, writing the result underneath the next coefficient.

6	2	–9	2	1
×		12		
	2			

Step 5

Add the numbers in this column.

6	2	–9	2	1
		+ 12		
	2	3		

Repeat Steps 4 and 5 until the last column has been completed.

The number in the lower-right cell is the value of the polynomial for the input value, often referred to as the **remainder**.

6	2	–9	2	1	
		12	18	120	
	2	3	20	121	$= f(6)$

EXAMPLE

1. Given $f(x)=x^3+x^2-22x-40$, evaluate $f(-2)$ using synthetic division.

–2	1	1	–22	–40

using the above process →

–2	1	1	–22	–40
		–2	2	40
	1	–1	–20	0

So $f(-2)=0$.

Note

In this example, the remainder is zero, so $f(-2)=0$.

This means $x^3+x^2-22x-40=0$ when $x=-2$, which means that $x=-2$ is a root of the equation. So $x+2$ must be a factor of the cubic.

We can use this to help with factorisation:

$$f(x)=(x+2)(\ q(x)\ ) \qquad \text{where } q(x) \text{ is a quadratic}$$

Is it possible to find the quadratic $q(x)$ using the table?

Trying the numbers from the bottom row as coefficients, we find:

$$\begin{aligned}&(x+2)(x^2-x-20)\\ &=x^3-x^2-20x+2x^2-2x-40\\ &=x^3-x^2-22x-40\\ &=f(x).\end{aligned}$$

–2	1	1	–22	–40
		–2	2	40
	1	–1	–20	0

So using the numbers from the bottom row as coefficients has given the correct quadratic. In fact, this method *always* gives the correct quadratic, making synthetic division a useful tool for factorising polynomials.

EXAMPLES

2. Show that $x-4$ is a factor of $2x^4-9x^3+5x^2-3x-4$.

$x-4$ is a factor $\Leftrightarrow$ $x=4$ is a root.

4	2	–9	5	–3	–4
		8	–4	4	4
	2	–1	1	1	0

Since the remainder is zero, $x=4$ is a root, so $x-4$ is a factor.

3. Given $f(x)=x^3-37x+84$, show that $x=-7$ is a root of $f(x)=0$, and hence fully factorise $f(x)$.

-7	1	0	-37	84
		-7	49	-84
	1	-7	12	0

Since the remainder is zero, $x=-7$ is a root.

Hence we have
$$\begin{aligned} f(x) &= x^3-37x+84 \\ &= (x+7)(x^2-7x+12) \\ &= (x+7)(x-3)(x-4). \end{aligned}$$

4. Show that $x=-5$ is a root of $2x^3+7x^2-9x+30=0$, and hence fully factorise the cubic.

-5	2	7	-9	30
		-10	15	-30
	2	-3	6	0

Since $x=-5$ is a root, $x+5$ is a factor.

$$2x^3+7x^2-9x+30=(x+5)(2x^2-3x+6)$$

This does not factorise any further since the quadratic has $b^2-4ac<0$.

Using synthetic division to factorise

In the examples above, we have been given a root or factor to help factorise polynomials. However, we can still use synthetic division if we do not know a factor or root.

Provided that the polynomial has an integer root, it will divide the constant term exactly. So by trying synthetic division with all divisors of the constant term, we will eventually find the integer root.

5. Fully factorise $2x^3+5x^2-28x-15$.

Numbers which divide -15: $\pm1, \pm3, \pm5, \pm15$.

Try $x=1$: $2(1)^3+5(1)^2-28(1)-15$
$$= 2+5-28-15 \neq 0.$$

Try $x=-1$: $2(-1)^3+5(-1)^2-28(-1)-15$
$$= -2+5+28-15 \neq 0.$$

Note
For ±1, it is simpler just to evaluate the polynomial directly, to see if these values are roots.

Try $x = 3$:

3	2	5	–28	–15
		6	33	15
	2	11	5	0

Since $x = 3$ is a root, $x - 3$ is a factor.

So $2x^3 + 5x^2 - 28x - 15 = (x-3)(2x^2 + 11x + 5)$
$= (x-3)(2x+1)(x+5)$.

Using synthetic division to solve equations

We can also use synthetic division to help solve equations.

EXAMPLE

6. Find the solutions of $2x^3 - 15x^2 + 16x + 12 = 0$.

Numbers which divide 12: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

Try $x = 1$: $2(1)^3 - 15(1)^2 + 16(1) + 12$
$= 2 - 15 + 16 + 12 \neq 0$.

Try $x = -1$: $2(-1)^3 - 15(-1)^2 + 16(-1) + 12$
$= -2 - 15 - 16 + 12 \neq 0$.

Try $x = 2$:

2	2	–15	16	12
		4	–22	–12
	2	–11	–6	0

Since $x = 2$ is a root, $x - 2$ is a factor:

$$2x^3 - 15x^2 + 16x + 12 = 0$$
$$(x-2)(2x^2 - 11x - 6) = 0$$
$$(x-2)(2x+1)(x-6) = 0$$

$x - 2 = 0$ or $2x + 1 = 0$ or $x - 6 = 0$
$x = 2$ $\quad x = -\frac{1}{2}$ $\quad x = 6$.

The Factor Theorem and Remainder Theorem

For a polynomial $f(x)$:

If $f(x)$ is divided by $x - h$ then the remainder is $f(h)$, and
$f(h) = 0 \Leftrightarrow x - h$ is a factor of $f(x)$.

As we saw, synthetic division helps us to write $f(x)$ in the form

$$(x-h)q(x) + f(h)$$

where $q(x)$ is called the **quotient** and $f(h)$ the **remainder**.

EXAMPLE

7. Find the quotient and remainder when $f(x) = 4x^3 + x^2 - x - 1$ is divided by $x + 1$, and express $f(x)$ as $(x+1)q(x) + f(h)$.

–1	4	1	–1	–1
		–4	3	–2
	4	–3	2	–3

The quotient is $4x^2 - 3x + 2$ and the remainder is –3, so

$$f(x) = (x+1)\left(4x^2 - 3x + 2\right) - 3.$$

10 Finding Unknown Coefficients

Consider a polynomial with some unknown coefficients, such as $x^3 + 2px^2 - px + 4$, where p is a constant.

If we divide the polynomial by $x - h$, then we will obtain an expression for the remainder in terms of the unknown constants. If we already know the value of the remainder, we can solve for the unknown constants.

EXAMPLES

1. Given that $x - 3$ is a factor of $x^3 - x^2 + px + 24$, find the value of p.

$x - 3$ is a factor $\Leftrightarrow$ $x = 3$ is a root.

3	1	–1	p	24
		3	6	$18 + 3p$
	1	2	$6 + p$	$42 + 3p$

Note
This is just the same synthetic division procedure we are used to.

Since $x = 3$ is a root, the remainder is zero:

$$\begin{aligned} 42 + 3p &= 0 \\ 3p &= -42 \\ p &= -14. \end{aligned}$$

2. When $f(x) = px^3 + qx^2 - 17x + 4q$ is divided by $x - 2$, the remainder is 6, and $x - 1$ is a factor of $f(x)$.

Find the values of p and q.

When $f(x)$ is divided by $x - 2$, the remainder is 6.

$$\begin{array}{c|cccc} 2 & p & q & -17 & 4q \\ & & 2p & 4p+2q & 8p+4q-34 \\ \hline & p & 2p+q & 4p+2q-17 & \boxed{8p+8q-34} \end{array}$$

Since the remainder is 6, we have:

$$\begin{aligned} 8p + 8q - 34 &= 6 \\ 8p + 8q &= 40 \\ p + q &= 5. \quad ① \end{aligned}$$

Since $x - 1$ is a factor, $f(1) = 0$:

$$\begin{aligned} f(1) &= p(1)^3 + q(1)^2 - 17(1) + 4q \\ &= p + q - 17 + 4q \\ &= p + 5q - 17 \end{aligned}$$

i.e. $p + 5q = 17.$ ②

Note
There is no need to use synthetic division here, but you could if you wish.

Solving ① and ② simultaneously, we obtain:

②−①: $4q = 12$

$q = 3.$

Put $q = 3$ into ①: $p + 3 = 5$

$p = 2.$

Hence $p = 2$ and $q = 3$.

11 Finding Intersections of Curves

We have already met intersections of lines and parabolas in this outcome, but we were mainly interested in finding equations of tangents

We will now look at how to find the actual points of intersection – and not just for lines and parabolas; the technique works for any polynomials.

EXAMPLES

1. Find the points of intersection of the line $y = 4x - 4$ and the parabola $y = 2x^2 - 2x - 12$.

To find intersections, equate:

$$\begin{aligned} 2x^2 - 2x - 12 &= 4x - 4 \\ 2x^2 - 6x - 8 &= 0 \\ x^2 - 3x - 4 &= 0 \\ (x+1)(x-4) &= 0 \\ x = -1 \quad \text{or} \quad x &= 4. \end{aligned}$$

Find the y-coordinates by putting the x-values into one of the equations:

when $x = -1$, $y = 4 \times (-1) - 4 = -4 - 4 = -8$,
when $x = 4$, $y = 4 \times 4 - 4 = 16 - 4 = 12$.

So the points of intersection are $(-1,-8)$ and $(4,12)$.

2. Find the coordinates of the points of intersection of the cubic $y = x^3 - 9x^2 + 20x - 10$ and the line $y = -3x + 5$.

To find intersections, equate:

$$\begin{aligned} x^3 - 9x^2 + 20x - 10 &= -3x + 5 \\ x^3 - 9x^2 + 23x - 15 &= 0 \\ (x-1)(x^2 - 8x + 15) &= 0 \\ (x-1)(x-3)(x-5) &= 0 \\ x = 1 \quad \text{or} \quad x = 3 \quad \text{or} \quad x &= 5. \end{aligned}$$

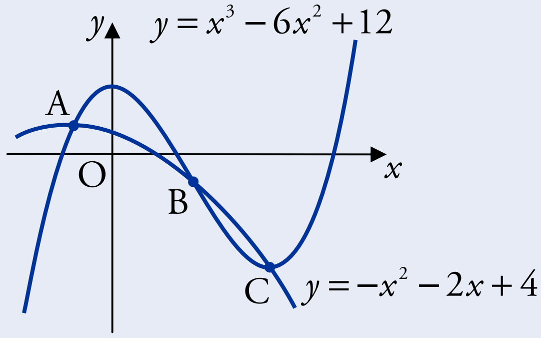
Remember
You can use synthetic division to help with factorising.

Find the y-coordinates by putting the x-values into one of the equations:

when $x = 1$, $y = -3 \times 1 + 5 = -3 + 5 = 2$,
when $x = 3$, $y = -3 \times 3 + 5 = -9 + 5 = -4$,
when $x = 5$, $y = -3 \times 5 + 5 = -15 + 5 = -10$.

So the points of intersection are $(1,2)$, $(3,-4)$ and $(5,-10)$.

3. The curves $y=-x^2-2x+4$ and $y=x^3-6x^2+12$ are shown below.



Find the x-coordinates of A, B and C, where the curves intersect.

To find intersections, equate:

$$\begin{aligned}-x^2-2x+4&=x^3-6x^2+12\\x^3-5x^2+2x+8&=0\\(x+1)(x^2-6x+8)&=0\\(x+1)(x-2)(x-4)&=0\\x=-1 \quad\text{or}\quad x=2 &\quad\text{or}\quad x=4.\end{aligned}$$

So at A, $x=-1$; at B, $x=2$; and at C, $x=4$.

Remember
You can use synthetic division to help with factorising.

4. Find the x-coordinates of the points where the curves $y=2x^3-3x^2-10$ and $y=3x^3-10x^2+7x+5$

To find intersections, equate:

$$\begin{aligned}2x^3-3x^2-10&=3x^3-10x^2+7x+5\\x^3-7x^2+7x+15&=0\\(x+1)(x^2-8x+15)&=0\\(x+1)(x-3)(x-5)&=0\\x=-1 \quad\text{or}\quad x=3 &\quad\text{or}\quad x=5.\end{aligned}$$

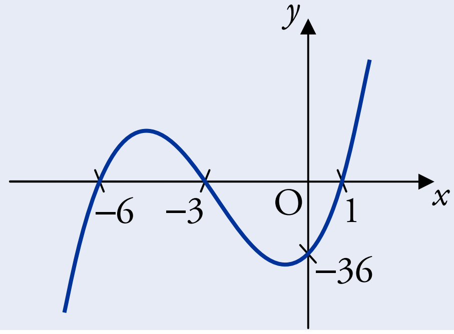
So the curves intersect where $x=-1,3,5$.

12 Determining the Equation of a Curve

Given the roots, and at least one other point lying on the curve, we can establish its equation using a process similar to that used when finding the equation of a parabola.

EXAMPLE

1. Find the equation of the cubic shown in the diagram below.



Step 1

Write out the roots, then rearrange to get the factors.

$$x = -6 \qquad x = -3 \qquad x = 1$$
$$x + 6 = 0 \qquad x + 3 = 0 \qquad x - 1 = 0.$$

Step 2

The equation then has these factors multiplied together with a constant, k.

$$y = k(x+6)(x+3)(x-1).$$

Step 3

Substitute the coordinates of a known point into this equation to find the value of k.

Using $(0, -36)$:

$$\begin{aligned} k(0+6)(0+3)(0-1) &= -36 \\ k(3)(-1)(6) &= -36 \\ -18k &= -36 \\ k &= 2. \end{aligned}$$

Step 4

Replace k with this value in the equation.

$$\begin{aligned} y &= 2(x+6)(x+3)(x-1) \\ &= 2(x+3)\left(x^2+5x-6\right) \\ &= 2\left(x^3+5x^2-6x+3x^2+15x-18\right) \\ &= 2x^3+16x^2+18x-36. \end{aligned}$$

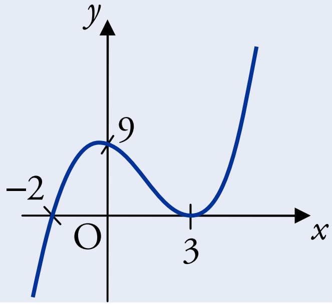
Repeated Roots

If a repeated root exists, then a stationary point lies on the x-axis.

Recall that a repeated root exists when two roots, and hence two factors, are equal.

EXAMPLE

2. Find the equation of the cubic shown in the diagram below.



$$x=-2 \qquad x=3 \qquad x=3$$
$$x+2=0 \qquad x-3=0 \qquad x-3=0.$$

So $y=k(x+2)(x-3)^2$.

Use $(0,9)$ to find k:

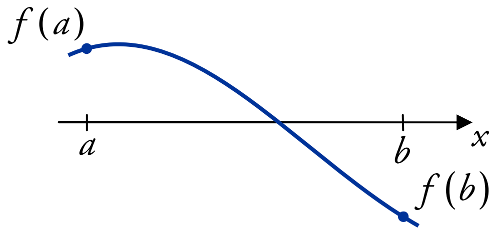
$$9=k(0+2)(0-3)^2$$
$$9=k\times 2\times 9$$
$$k=\tfrac{1}{2}.$$

So
$$\begin{aligned} y&=\tfrac{1}{2}(x+2)(x-3)^2\\ &=\tfrac{1}{2}(x+2)\left(x^2-6x+9\right)\\ &=\tfrac{1}{2}\left(x^3-6x^2+9x+2x^2-12x+18\right)\\ &=\tfrac{1}{2}x^3-2x^2-\tfrac{3}{2}x+9.\end{aligned}$$

Note
$x=3$ is a repeated root, so the factor $(x-3)$ appears twice in the equation.

13 Approximating Roots

Polynomials have the special property that if $f(a)$ is positive and $f(b)$ is negative then f must have a root between a and b.



We can use this property to find approximations for roots of polynomials to any degree of accuracy by repeatedly "zooming in" on the root.

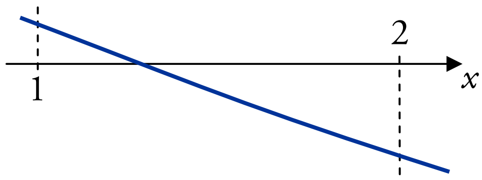
EXAMPLE

Given $f(x)=x^3-4x^2-2x+7$, show that there is a real root between $x=1$ and $x=2$. Find this root correct to two decimal places.

Evaluate $f(x)$ at $x=1$ and $x=2$:

$$f(1)=1^3-4(1)^2-2(1)+7=2$$

$$f(2)=2^3-4(2)^2-2(2)+7=-5$$



Since $f(1)>0$ and $f(2)<0$, $f(x)$ has a root between these values.

Start halfway between $x=1$ and $x=2$, then take little steps to find a change in sign:

$$f(1{\cdot}5)=-1{\cdot}625<0$$

$$f(1{\cdot}4)=-0{\cdot}896<0$$

$$f(1{\cdot}3)=-0{\cdot}163<0$$

$$f(1{\cdot}2)=0{\cdot}568>0.$$

Since $f(1{\cdot}2)>0$ and $f(1{\cdot}3)<0$, the root is between $x=1{\cdot}2$ and $x=1{\cdot}3$.

Start halfway between $x=1{\cdot}2$ and $x=1{\cdot}3$:

$$f(1{\cdot}25)=0{\cdot}203125>0$$

$$f(1{\cdot}26)=0{\cdot}129976>0$$

$$f(1{\cdot}27)=0{\cdot}056783>0$$

$$f(1{\cdot}28)=-0{\cdot}016448<0.$$

Since $f(1{\cdot}27)>0$ and $f(1{\cdot}28)<0$, the root is between these values.

Finally, $f(1.275)=0.020171875>0$. Since $f(1{\cdot}275)>0$ and $f(1{\cdot}28)<0$, the root is between $x=1{\cdot}275$ and $x=1{\cdot}28$.

Therefore the root is $x=1{\cdot}28$ to 2 d.p.