Advanced Higher Mathematics

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Unit 4

Further Proof, Summations, Number Theory.

Outcome 5 — Further Number Theory and Further Methods of Proof.

Further Mathematical Proof

Negation of a Statement:-

Statement : A rhombus has four equal sides. (can be true or false)
Negation : A rhombus does not have four equal sides.

If a statement is **true** then the negation is **false**.
If a statement is **false** then the negation is **true**.

Given that p represents a statement, the negation is written $\sim p$ (reads "not p"), and is such that if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

Examples Negate the following statements:-

1. "All cats have tails".

   p : All cats have tails.

   Since the given statement is taken to be true for every cat, the negation must assert that at least one cat has no tail and so:-

   $\sim p$: Some cats do not have tails.

2. "Some pilots are women".

   p : Some pilots are women.

   Since the given statement asserts that there is at least one pilot who is a woman, the negation must assert that all pilots are not women and so:-

   $\sim p$: No pilots are women.

Exercise 1

1. Which of the following is a negation of "All boys are adventurous" ?
   (a) No boys are adventurous.
   (b) All boys are unadventurous.
   (c) Some boys are not adventurous.
   (d) No boys are unadventurous.

2. Which of these is a negation of "No visitors may walk on the grass" ?
   (a) All visitors may walk on the grass.
   (b) Some visitors may not walk on the grass.
   (c) All visitors may not walk on the grass.
   (d) Some visitors may walk on the grass.

3. Write down the negation of each of the following statements:-
   (a) For all real x, x^2 is positive.
   (b) Some pupils find mathematics difficult.
   (c) No dogs like cats.
   (d) There exists a positive integer x such that $x + 3 > 0$.
   (e) Every parallelogram has half turn symmetry.
   (f) No schoolboy lies.
   (g) A number which has zero in the units place is divisible by five.
   (h) All numbers of the form $2^n - 1$, (n an integer), are prime.

The Converse of a Statement

Statement : **If** a triangle is right–angled, **then** the square of the hypotenuse is equal to the sum of the squares on the other two sides.

Converse : **If** the square on the longest side of a triangle is equal to the sum of the squares on the other two sides, **then** the triangle is right angled.

If $p \Rightarrow q$ **then** the converse is $q \Rightarrow p$. ($\Rightarrow$ means "implies")

In the cases of a statement and its converse:-

(i) the statement may be true and the converse false.
(ii) the statement may be be false and the converse true.
(iii) both may be false or both may be true.

Exercise 2 State the **converse** of each of the following, and show by a counter example that the converse is false.

1. (a) If a number ends in 0, it is divisible by 5.
(b) All primes greater than 2 are odd numbers.
(c) If a quadrilateral is a square, its diagonals intersect at right angles.
(d) $x = 3 \Rightarrow x^2 = 9$.
(e) If two numbers are odd then their sum is even.
(f) If two integers are even, then their product is even.

Equivalent Statements

If a statement and its converse are true then the implication can be replaced by the two-way implication sign $\Leftrightarrow$.

If $p \Rightarrow q$ and $q \Rightarrow p$ are both true then we can write $p \Leftrightarrow q$.
This is sometimes read as " p (is true) **if and only if** q (is true)", or "p iff q".

If $p \Rightarrow q$, we say that p is a **sufficient condition** for q ; if $q \Rightarrow p$, we say that p is a **necessary condition** for q.

Examples In each of the following, say whether the first statement is:-

(a) a necessary condition, (b) a sufficient condition,
(c) both necessary and sufficient, (d) neither,

for the second condition condition.

1. p: John plays the piano. q: John is a concert pianist.

Since $q \Rightarrow p$, the first statement is a necessary condition for the second as John must be able to play the piano to be concert pianist, but since $p \not\Rightarrow q$, it is not a sufficient condition.

2. p: ABCD is a rhombus q: the diagonals of ABCD bisect each other.

Since $p \Rightarrow q$, the first statement is a sufficient condition for the second as the diagonals of a rhombus bisect each other.
Since $q \not\Rightarrow p$, it is not a necessary condition.

<u>Exercise 3</u>

1. For each of these, say whether the first statement is:-
   (i) a necessary condition, (ii) a sufficient condition,
   (iii) both necessary and sufficient, (iv) neither,
   for the second condition condition.
   (a) p: there are more than 8 people in this room
       q: there are 9 people in this room.
   (b) p: ABCD is a parallelogram
       q: the diagonals of ABCD are perpendicular.
2. Which of the following statements are necessary or/and sufficient for the statement q: "natural number n is divisible by 6" to be correct ?
   (a) p: n is divisible by 3 (b) p: n is divisible by 9
   (c) p: n is divisible by 12 (d) p: n^2 is divisible by 12
   (e) p: $n = 384$ (f) p: n is even and divisible by 3
   (g) $n = m(m + 1)(m + 2)$, where m is some natural number.

Contrapositive of a Statement

If $p \Rightarrow q$ then the contrapositive is $\sim q \Rightarrow \sim p$.
i.e. The negation of q implies the negation of p.
If $p \Rightarrow q$ is true then $\sim q \Rightarrow \sim p$ must also be true.
Similarly, if $p \Rightarrow q$ is false then $\sim q \Rightarrow \sim p$ must also be false.
This is an important logical method of proving by indirect proof.

Method of Proof

Proof using Contrapositive

1. Prove that if x and y are integers and $xy = 100$, then either $x \leq 10$ or $y \leq 10$.
   p : x and y are integers and $xy = 100$,
   q : either $x \leq 10$ or $y \leq 10$.
   $\sim p$: x and y are integers and $xy \neq 100$
   $\sim q$: $x > 10$ <u>and</u> $y > 10$, then
   Proof of contrapositive:- (i.e. show $\sim q \Rightarrow \sim p$)
   $x > 10$ and $y > 10 \Rightarrow xy > 100 \Rightarrow xy \neq 100$.
   $\Rightarrow$ The contrapositive is true and hence the statement is true.
2. Prove that if 7 is a factor of n^2 then 7 is a factor of n.
   p : 7 is a factor of n^2 q: 7 is a factor of n.
   $\sim p$: 7 is a not factor of n^2 $\sim q$: 7 is a not factor of n.
   Proof of contrapositive:-
   7 is a not factor of n $\Rightarrow$ $n = 7m + t$ for some integers m and t.
   $\Rightarrow$ $n^2 = 49m^2 + 14mt + t^2$
   $\Rightarrow$ $n^2 = 7(7m^2 + 2mt) + t^2 = 7k +$ remainder
   $\Rightarrow$ 7 is not a factor of n^2.
   The contrapositive is true and hence the statement is true.

Proof by Contradiction

This is an extension of work in Mathematics 2(AH) Learning Outcome 5. To prove a theorem by this method we assume the theorem does not hold, then show that this assumption leads to a contradiction.

Examples

1. Prove that $\sqrt{2}$ is irrational.

   Suppose the opposite is true. i.e. $\sqrt{2}$ is rational.

   $\Rightarrow \quad \sqrt{2} = \dfrac{m}{n}$ where m and n have no common factor

   $\Rightarrow \quad 2 = \dfrac{m^2}{n^2} \Rightarrow m^2 = 2n^2 \Rightarrow m^2$ is even $\Rightarrow m$ is even $= 2k$ ($k \in \mathbb{Z}$)

   $\Rightarrow \quad 2n^2 = 4k^2 \Rightarrow n^2 = 2k^2 \Rightarrow n^2$ is even $\Rightarrow n$ is even

   $\Rightarrow \quad m$ and n have a common factor.

   But m and n have no common factor.

   Therefore $\sqrt{2}$ cannot be written in the form $\dfrac{m}{n}$

   This contradicts our assumption that $\sqrt{2}$ is rational.

   Hence $\sqrt{2}$ is irrational (by the method of contradiction).

2. If x and y are integers and xy is an odd integer, prove that x and y must both be odd.

   Assume the opposite.

   i.e. assume that xy is odd and at least one of x and y is even.

   If x is even $\Rightarrow x = 2z$, z is an integer.

   $\Rightarrow xy = 2zy$ (an even integer)

   $\Rightarrow xy$ is even.

   Since this is a contradiction, $\Rightarrow$ the initial assumption is wrong.

   If x and y are integers such that xy is an odd integer, then x and y must both be odd.

Exercise 4

1. Prove, using the contrapositive,
   (a) that if x and y are integers and xy is odd, then both x and y are odd.
   (b) that every prime number greater than 3 is of the form $6n \pm 1$, where n is a positive integer.
   (c) that if n is a natural number such that n^2 is even, then n is even.

2. Prove, by contradiction,
   (a) that if x and y are integers such that $x + y$ is odd, then one of them must be odd and one must be even.
   (b) that if x and y are real numbers such that $x + y$ is irrational, then at least one of x, y is irrational.
   (c) that if m and n are integers such that mn^2 is even, then at least one of m or n is even.
   (d) that if $\sin\theta \neq 0$, then $\theta \neq k\pi$ for any integer k.

Proof by Induction

This is an extension of work in Mathematics 2(AH) Learning Outcome 5. We are now going to look at further examples of this method of proof.

Examples 1. Prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

We know that $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2$

Proof :- for $n = 1$, $\sum_{r=1}^{n} r^2 = \sum_{r=1}^{1} r^2 = 1^2 = 1$

and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$ => true for $n = 1$

Assume true for $n = k$, $k \geq 1$ i.e. $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$

=> prove true for $n = k+1$ i.e. $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$

$$\begin{aligned}\sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^{k} r^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right] \text{ by factorisation} \\ &= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3)\end{aligned}$$

Hence if true for k, it is true for $n = k + 1$
True for $n = 1$ => True for $n = 2$ since $k \geq 1$
True for $n = 2$ => True for $n = 3$ and so on for all n
Hence true for all n, by induction.

2. Prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left[\sum_{r=1}^{n} r\right]^2$

We know that $\sum_{r=1}^{n} r^3 = 1^3 + 2^2 + 3^3 + \ldots + (n-1)^3 + n^3$

Proof :- for $n = 1$, $\sum_{r=1}^{n} r^3 = \sum_{r=1}^{1} r^3 = 1^3 = 1$

and $\frac{1}{4}n^2(n+1)^2 = \frac{1}{4} \times 1^2 \times 2^2 = 1$ => True for $n = 1$

Assume true for $n = k$, $k \geq 1$ i.e. $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

Hence prove true for $n = k + 1$ i.e. $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$

$$\Rightarrow \quad \sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right] \text{ by factorisation}$$

$$= \frac{1}{4}(k+1)^2\left(k^2 + 4k + 4\right)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

Hence if true for k, it is also true for $n = k + 1$
True for $n = 1$ => True for $n = 2$ since $k \geq 1$
True for $n = 2$ => True for $n = 3$ etc. for all n.
and so true for all n, by induction.

3. For all integers $n \geq 10$, $n^2 \geq 10n$.

Proof :- for $n = 10$, $n^2 = 100$ and $10n = 100$
=> True for $n = 10$.

Assume true for $n = k$, $k \geq 10$ i.e. $k^2 \geq 10k$.
Prove true for $n = k + 1$ also i.e. $(k+1)^2 \geq 10(k+1)$.

$$\Rightarrow \quad (k+1)^2 = k^2 + 2k + 1 \geq 10k + 2k + 1$$

$$\geq 10(k+1) + (2k - 9)$$

$$\geq 10(k+1)$$

since $2k - 9 \geq 11$

Hence if true for k, it is true for $n = k + 1$

True for $n = 10$ => True for $n = 11$ since $k \geq 10$
True for $n = 11$ => True for $n = 12$
and so on true for all values of n, by induction.

4. Show that $4^n + 6n - 1$ is divisible by 9 for all $n \geq 1$

Proof :- for $n = 1$,

$4^n + 6n - 1 = 4 + 6 - 1 = 9$, divisible by 9.

True for $n = 1$

Assume true for $n = k$, $k \geq 1$

(We want to show that $4^k + 6k - 1$ is divisible by 9

$\Rightarrow 4^{k+1} + 6(k + 1) - 1$ is divisible by 9) $(= 9p)$, $p \in W$

$$4^{k+1} + 6(k + 1) - 1 = 4.4^k + 6k + 5$$
$$= 4(4^k + 6k - 1) - 18k + 9$$

But since $4^k + 6k - 1$ is divisible by 9 $(= 9p)$, $p \in W$

$$\Rightarrow 4^{k+1} + 6(k + 1) - 1 = 4.9p - 18k + 9$$
$$= 9(4p - 2k + 1) \quad (= 9t) \; t \in W$$

$\Rightarrow 4^{k+1} + 6(k + 1) - 1$ is divisible by 9. $(= 9t)$ $t \in W$

Hence if true for k, it is true for $n = k + 1$.

True for $n = 1 \Rightarrow$ True for $n = 2$ since $k \geq 1$

True for $n = 2 \Rightarrow$ True for $n = 3$

and so on for all values of n.

5. Prove that S_n of the series $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots\ldots = \frac{n}{n+1}$

Proof :- for $n = 1$, $S_1 = \frac{1}{1\times2} = \frac{1}{2}$, $\frac{n}{n+1} = \frac{1}{2}$ (true)

for $n = 2$, $S_2 = S_1 + \frac{1}{2\times3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, $\frac{n}{n+1} = \frac{2}{3}$

True for $n = 1$ and $n = 2$

Assume it is true for $n = k$, i.e. $S_k = \frac{k}{k+1}$

$$S_{k+1} = S_k + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$
$$= \frac{k+1}{(k+1)+1}$$

Hence if true for k, it is true for $n = k + 1$

True for $n = 1 \Rightarrow$ True for $n = 2$ since $k \geq 1$

True for $n = 2 \Rightarrow$ True for $n = 3$

and so on for all values of n.

Exercise 5 Prove the following by induction :-

1. $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

2. $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$.

3. S_n of the series $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots\ldots = \frac{n}{2n+1}$

4. S_n of the series $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \ldots\ldots = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$

5. Use the results of $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to prove by direct method

   (a) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

   (b) $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

6. $n^3 + 3n^2 - 10n$ is divisible by 3.

7. $7^n + 4^n + 1^n$ is divisible by 6.

8. For all integers $n > 2$, $2^n > 2n$.

9. For all integers $n \geq 4$, $3^n > n^3$.

Number Theory

The Division Algorithm

If a is a non-negative integer and b a positive integer, then there exists unique non-negative integers q and r such that

$$a = bq + r \quad \text{and } 0 \le r < b$$

Proof :- On the real number line, the integers $a, a-b, a-2b, a-3b, \ldots..$ form an decreasing sequence of integers.

Since only finitely many of these are ≥ 0, there is a unique integer $q \ge 0$ for which

$$a - (q+1)b < 0 \le a - bq$$

and so $$0 \le a - bq < b$$

If we write $r = a - bq$, then $a = bq + r$ with $0 \le r < b$.

Thus we have found non-negative integers q and r for which
$a = bq + r$ and $0 \le r < b$ both hold.

To show that q and r are unique, suppose that
$$a = bq_1 + r_1 \quad \text{and } 0 \le r_1 < b.$$
Then $r_1 = a - bq_1$ and $0 \le a - bq_1 < b$.
It follows that $a - (q_1 + 1)b < 0 \le a - bq_1$ so that q_1 is the integer determined above and $r_1 = a - bq = r$.
Thus the theorem is proved.

<u>Examples</u>

1. $a = 193$ and $b = 17$

   $193 = 11.17 + 6$ $\qquad q = 11,\ r = 6$

2. $a = 581$ and $b = 23$

   $581 = 25.23 + 6$ $\qquad q = 25,\ r = 6$

<u>Exercise 6</u> Use the division identity for the following

1. $a = 75$ and $b = 12$
2. $a = 327$ and $b = 13$
3. $a = 392$ and $b = 19$

If $r = 0$ then we say that b is a divisor of a.

The notation used is $b \mid a$ which means "b is the **divisor** of a".

contd...

Euclidean Algorithm

The Euclidean Algorithm is used to find the greatest common divisor (G.C.D) of 2 or more positive integers where this cannot be done simply.

For integers a and b,

$$a = bq_1 + r_1 \quad \text{and } 0 \le r_1 < b$$
$$b = r_1q_2 + r_2 \quad \text{and } 0 \le r_2 < r_1$$
$$r_1 = r_2q_3 + r_3 \quad \text{and } 0 \le r_3 < r_2$$

and so on until

$$r_{n-2} = r_{n-1}q_n + r_n$$
$$r_{n-1} = r_nq_{n+1} + 0. \qquad \text{(i.e. } r \text{ eventually becomes 0)}$$

To find the G.C.D. for <u>small numbers</u>, you use factorisation as follows :-

<u>**Example**</u> Find the G.C.D. of 15 and 24

$15 = 3 \times 5$ and $24 = 2^3 \times 3$

so the G.C.D. of 15 and 24 is 3

<u>Notation</u> $(15, 24) = 3$ (i.e. G.C.D. of 15 and 24 is 3)

To find the G.C.D. for <u>large numbers</u>, use the Euclidean Algorithm.

<u>**Examples**</u> Use the Euclidean Algorithm to find the G.C.D. of (1147, 851).

Use repeated application of the division identity until $r = 0$.
=> The last non-zero remainder is the G.C.D.

$$1147 = 1 \times 851 + 296$$
$$851 = 2 \times 296 + 259$$
$$296 = 1 \times 259 + \underline{37}$$
$$259 = 7 \times \underline{\underline{37}} + 0$$

Hence $(1147, 851) = \underline{\underline{37}}$

<u>**Exercise 7**</u>

1. Find the G.C.D. of

   (i) (15, 27) (ii) (16, 42) (iii) (72, 108)

2. Use the Euclidean Algorithm to find the G.C.D. of

   (i) (1219, 901) (ii) (4277, 2821) (iii) (5213, 2867)

Expressing the G.C.D. of two Positive Integers as a Linear Combination of the two Integers.

Having found the G.C.D. of two positive integers a and b, it is possible, by working backwards, to express the divisor (d) in terms of the two integers in the form of a linear combination.

i.e. $d = xa + yb$ where x and y are integers.

Example

1. (a) Use the Euclidean Algorithm to find the G.C.D.of (1147, 851).

   (b) Hence find the integers x and y to write this G.C.D. in the form $x.1147 + y.851$.

(a)
$$\begin{aligned} 1147 &= 1 \times 851 + 296 && (1)\\ 851 &= 2 \times 296 + 259 && (2)\\ 296 &= 1 \times 259 + 37 && (3)\\ 259 &= 7 \times 37 + 0 && (4) \end{aligned}$$

Hence $(1147, 851) = 37$

(b)
$$\begin{aligned} &\text{From (3)} && 37 = 296 - 1 \times 259\\ &\text{From (2)} && \quad = 296 - 1 \times (851 - 2 \times 296)\\ &&& \quad = 296 - 1 \times 851 + 2 \times 296\\ &&& \quad = 3 \times 296 - 1 \times 851\\ &\text{From (1)} && \quad = 3 \times (1147 - 1 \times 851) - 1 \times 851\\ &&& \quad = 3 \times 1147 - 3 \times 851 - 1 \times 851\\ &&& \quad = 3 \times 1147 - 4 \times 851 \end{aligned}$$

$x = 3$ and $y = -4$

Exercise 8

1. Express the G.C.D.'s found in Exercise 7 Question 2 as a linear combination of the original numbers.

2. (a) Use the Euclidean Algorithm to find the G.C.D.of (7293, 798).

   (b) Hence find the integers x and y to write this G.C.D. in the form $x.7293 + y.798$.

Expressing Base 10 integers in other Bases

Examples

1. Express 235_{ten} in the base 6.

$$235 = 6 \times 39 + 1$$
$$39 = 6 \times 6 + 3$$
$$6 = 6 \times 1 + 0$$
$$1 = 6 \times 0 + 1$$

Reading the remainders in reverse gives 1031_{six}

or
$$\begin{aligned} 235 &= 6 \times 39 + 1 \\ &= 6 \times (6 \times 6 + 3) + 1 \\ &= 6 \times 6 \times 6 + 3 \times 6 + 1 \\ &= 1 \times 6^3 + 0 \times 6 + 3 \times 6 + 1 \\ &= 1031_{six} \end{aligned}$$

2. Express 423_{ten} in the base 8.

$$423 = 8 \times 52 + 7$$
$$52 = 8 \times 6 + 4$$
$$6 = 8 \times 0 + 6$$

Reading the remainders in reverse gives 647_{eight}

or
$$\begin{aligned} 423 &= 8 \times 52 + 7 \\ &= 8 \times (8 \times 6 + 4) + 7 \\ &= 6 \times 8 \times 8 + 4 \times 8 + 7 \\ &= 6 \times 8^2 + 4 \times 8 + 7 \\ &= 647_{eight} \end{aligned}$$

Exercise 9

1. Express 81 in the base 2.
2. Express 579 in the base 5.
3. Express 1064 in the base 7.
4. Express 15287 in the base 9.

Answers

Exercise 1

1. (c) 2. (d)

3. (a) For some real x, x^2 is not positive.
   (b) No pupils find mathematics difficult.
   (c) Some dogs like cats.
   (d) There is no positive integer x such that $x + 3 > 0$.
   (e) Some parallelograms do not have half turn symmetry.
   (f) Some school boys lie.
   (g) Some numbers withh a zero in the units place are not divisible by 5.
   (h) Some numbers of the form $2^n - 1$, (n an integer), are not prime.

Exercise 2

1. (a) If a number is divisible by 5, it ends in zero. (e.g. 15)
   (b) If a number is odd, it is aprime number greater than 2. (e.g. 21)
   (c) If the diagonals of a quadrilateral intersect at right angles, the quadrilateral is a square. (e.g. a rhombus)
   (d) If $x^2 = 9$, $x = 3$ (e.g. $x = 3$)
   (e) If the sum of two numbers is even, => the numbers are odd. (6 & 4)
   (f) If the product of two numbers is even, the numbers are even. (6 & 5)

Exercise 3

1. necessary 2. neither

3. (a) necessary (b) none of these (c) sufficient
   (d) necessary and sufficient (e) sufficient
   (f) necessary and sufficient (g) neither (h) sufficient

Exercise 4 All proofs

Exercise 5 All Proofs

Exercise 6

1. $75 = 6.12 + 3$ 2. $327 = 25.13 + 2$ 3. $392 = 20.19 + 12$

Exercise 7

1. (i) 3 (ii) 2 (iii) 36 2. (i) 53 (ii) 91 (iii) 1

Exercise 8

1. (a) $53 = 3 \times 1219 - 4 \times 901$ (b) $91 = 2 \times 4277 - 3 \times 2821$
   (c) $1 = -952 \times 5213 + 1731 \times 2867$

2. (a) 3 (b) $x = -115, y = 1051$

Exercise 9

1. 1010001_{two} 2. 4304_{five} 3. 3050_{seven} 4. 22865_{nine}

TJ3. p.85 Ex5 Nos 1→4, 6→9 Proof by Induction

(1) Prove $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$, $n \geq 1$, $n \in \mathbb{N}$.

$n=1$: LHS $\sum_{r=1}^{1} r(r+1)$ RHS $\frac{1}{3}(1)(1+1)(1+2)$

$= 1(1+1)$ $= \frac{1}{3} \times 1 \times 2 \times 3$

$= 1 \times 2$ $= \frac{1}{3} \times 6$

$= 2$ $= 2$

$\therefore$ True for $n=1$.

Assume true for $n=k$: $\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$ ------ (1).

Want to prove: $\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)(k+3)$ ------ (2)

$n=k+1$:

$$\sum_{r=1}^{k+1} r(r+1) = \sum_{r=1}^{k} r(r+1) + (k+1)(k+2)$$

$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$ ---- using (1)

$= \frac{1}{3}(k+1)(k+2)\left[k+3\right]$ ---- (taking $\frac{1}{3}(k+1)(k+2)$ as a common factor).

$= \frac{1}{3}(k+1)(k+2)(k+3)$ which is (2), as required.

i.e. true for $n=k+1$.

Summary True for $n=1$ and truth of $n=k \Rightarrow$ true for $n=k+1$
ie. true for $n=1 \Rightarrow$ true for $n=2$
true for $n=2 \Rightarrow$ true for $n=3$
etc.
ie. true for $n \geq 1$, $n \in \mathbb{N}$. *

[* This kind of summary at the end will be omitted for the remaining solutions, but should really be written each time.]

TJ3. p.85 Ex5 Nos 1→4, 6→9 Proof by Induction

① Prove $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$, $n \geqslant 1$, $n \in \mathbb{N}$.

$n=1$: LHS $\sum_{r=1}^{1} r(r+1)$ RHS $\frac{1}{3}(1)(1+1)(1+2)$

$= 1(1+1)$ $= \frac{1}{3} \times 1 \times 2 \times 3$

$= 1 \times 2$ $= \frac{1}{3} \times 6$

$= 2$ $= 2$

∴ True for $n=1$.

Assume true for $n=k$: $\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$ ---- (1).

Want to prove: $\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)(k+3)$ ----- (2)

$n=k+1$:

$$\sum_{r=1}^{k+1} r(r+1) = \sum_{r=1}^{k} r(r+1) + (k+1)(k+2)$$

$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$ ---- using (1)

$= \frac{1}{3}(k+1)(k+2)\left[k+3\right]$ ---- (taking $\frac{1}{3}(k+1)(k+2)$ as a common factor).

$= \frac{1}{3}(k+1)(k+2)(k+3)$ which is (2), as required.

i.e. true for $n=k+1$.

Summary: True for $n=1$ and truth of $n=k \Rightarrow$ true for $n=k+1$
ie. true for $n=1 \Rightarrow$ true for $n=2$
true for $n=2 \Rightarrow$ true for $n=3$
etc.
ie. true for $n \geqslant 1$, $n \in \mathbb{N}$. *

[* This kind of summary at the end will be omitted for the remaining solutions, but should really be written each time.]

(2) Prove $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$, $n \geq 1$, $n \in \mathbb{N}$.

$\underline{n=1}$: $\underline{\text{LHS}}$ $\sum_{r=1}^{1} r(r+1)(r+2)$ $\quad\quad$ $\underline{\text{RHS}}$ $\frac{1}{4}(1)(1+1)(1+2)(1+3)$

$= 1(1+1)(1+2)$ $\quad\quad$ $= \frac{1}{4} \times 1 \times 2 \times 3 \times 4$

$= 1 \times 2 \times 3$ $\quad\quad$ $= \frac{1}{4} \times$

$= 6$ $\quad\quad$ $= 6$

$\therefore$ True for $n=1$.

Assume true for $n=k$: i.e. $\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$, $k \geq 1$, $k \in \mathbb{N}$. ------(1)

Want to prove: $\sum_{r=1}^{k+1} r(r+1)(r+2) = \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ ------(2)

$\underline{n=k+1}$: $\sum_{r=1}^{k+1} r(r+1)(r+2) = \sum_{r=1}^{k} r(r+1)(r+2) + (k+1)(k+2)(k+3)$

$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$ --using (1)

$= \frac{1}{4}(k+1)(k+2)(k+3)\left[k+4\right]$ -- $\frac{1}{4}(k+1)(k+2)(k+3)$ as common factor

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ which is (2) as req^{d}.

i.e. true for $n=k+1$.

[then usual summary]

(2) Prove $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$, $n \geq 1$, $n \in \mathbb{N}$.

$\underline{n=1}$: $\underline{\text{LHS}}$ $\sum_{r=1}^{1} r(r+1)(r+2)$

$= 1(1+1)(1+2)$

$= 1 \times 2 \times 3$

$= 6$

$\underline{\text{RHS}}$ $\frac{1}{4}(1)(1+1)(1+2)(1+3)$

$= \frac{1}{4} \times 1 \times 2 \times 3 \times 4$

$= \frac{1}{4} \times$

$= 6$

$\therefore$ True for $n=1$.

Assume true for $n=k$: i.e. $\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$, $k \geq 1$, $k \in \mathbb{N}$. ---(1)

Want to prove: $\sum_{r=1}^{k+1} r(r+1)(r+2) = \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ -----(2)

$\underline{n=k+1}$: $\sum_{r=1}^{k+1} r(r+1)(r+2) = \sum_{r=1}^{k} r(r+1)(r+2) + (k+1)(k+2)(k+3)$

$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$ ... using (1)

$= \frac{1}{4}(k+1)(k+2)(k+3)[k+4]$ -- $\frac{1}{4}(k+1)(k+2)(k+3)$ as common factor

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ which is (2) as req^{d}.

i.e. true for $n=k+1$.

[then usual summary]

③ S_n of $\underbrace{\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots\cdots}_{n \text{ terms}}$

can be written as $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$

i.e. Prove $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}=\frac{n}{2n+1}$, $n\geq1, n\in\mathbb{N}$

$\underline{n=1}$ LHS $\sum_{r=1}^{1}\frac{1}{(2r-1)(2r+1)}$ RHS $\frac{n}{2n+1}$

$=\frac{1}{(2\times1-1)(2\times1+1)}$ $=\frac{1}{2\times1+1}$

$=\frac{1}{1\times3}$ $=\frac{1}{2+1}$

$=\frac{1}{3}$ $=\frac{1}{3}$

i.e. true for $n=1$.

Assume true for $n=k$: i.e. $\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}=\frac{k}{2k+1}$, $k\geq1, k\in\mathbb{N}$ ----(1)

Want to prove: $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{k+1}{2(k+1)+1}=\frac{k+1}{2k+3}$, $k\geq1, k\in\mathbb{N}$ ---(2)

$\underline{n=k+1}$: $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$

$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$ ----------using (1)

$=\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$=\frac{k+1}{2k+3}$ which is (2), as required. i.e. true for $n=k+1$ (etc)

③ S_n of $\underbrace{\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots}_{n \text{ terms}}$

can be written as $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$

i.e. Prove $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}=\frac{n}{2n+1}$, $n\geq 1, n\in\mathbb{N}$

$n=1$ LHS $\sum_{r=1}^{1}\frac{1}{(2r-1)(2r+1)}$ RHS $\frac{n}{2n+1}$

$=\frac{1}{(2\times1-1)(2\times1+1)}$ $=\frac{1}{2\times1+1}$

$=\frac{1}{1\times3}$ $=\frac{1}{2+1}$

$=\frac{1}{3}$ $=\frac{1}{3}$

ie. true for $n=1$.

Assume true for $n=k$: i.e. $\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}=\frac{k}{2k+1}$, $k\geq1, k\in\mathbb{N}$ ---(1)

Want to prove: $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{k+1}{2(k+1)+1}=\frac{k+1}{2k+3}$, $k\geq1, k\in\mathbb{N}$ ---(2)

$n=k+1$: $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$

$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$ ---------using (1)

$=\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$=\frac{k+1}{2k+3}$ which is (2), as required. ie. true for $n=k+1$ (etc)

④ S_n of the series $\underbrace{\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\cdots\cdots}_{n \text{ terms}}$ can be written as:

$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$$

i.e. Prove $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$, $n\geq1$, $n\in\mathbb{N}$ ---(1).

$\underline{n=1}$ $\underline{\text{LHS}}$ $\displaystyle\sum_{r=1}^{1}\frac{1}{r(r+1)(r+2)}$

$$=\frac{1}{1\times2\times3}$$

$$=\frac{1}{6}$$

$\underline{\text{RHS}}$ $\displaystyle\frac{1}{4}-\frac{1}{2(1+1)(1+2)}$

$$=\frac{1}{4}-\frac{1}{2\times2\times3}$$

$$=\frac{1}{4}-\frac{1}{12}$$

$$=\frac{3}{12}-\frac{1}{12}$$

$$=\frac{2}{12}$$

$$=\frac{1}{6}$$

i.e. true for $n=1$.

Assume true for $n=k$: i.e. $\displaystyle\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}$, $k\geq1$, $k\in\mathbb{N}$ ---(1)

Want to prove: $\displaystyle\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(k+1+1)(k+1+2)}$

$$=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}, \quad k\geq1, k\in\mathbb{N} \ldots (2)$$

$\underline{n=k+1}$: $\displaystyle\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \quad \text{[using (1)]}$$

$$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{2}{2(k+1)(k+2)(k+3)}$$

$$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}\left[1-\frac{2}{(k+3)}\right]$$

$$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}\left[\frac{k+3}{k+3}-\frac{2}{k+3}\right]$$

$$=\frac{1}{4}-\frac{1}{2\cancel{(k+1)}(k+2)}\left[\frac{\cancel{(k+1)}}{(k+3)}\right]=\frac{1}{4}-\frac{1}{2(k+2)(k+3)} \text{ --- (2) as required.}$$

i.e. true for $n=k+1$ (etc).

④ S_n of the series $\underbrace{\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \cdots\cdots}_{n \text{ terms}}$ can be written as:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$$

i.e. Prove $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$, $n \geq 1$, $n \in \mathbb{N}$ --- (1).

$n=1$ LHS $\displaystyle\sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)}$ RHS $\displaystyle\frac{1}{4} - \frac{1}{2(1+1)(1+2)}$

$$= \frac{1}{1\times2\times3} \qquad = \frac{1}{4} - \frac{1}{2\times2\times3}$$

$$= \frac{1}{6} \qquad = \frac{1}{4} - \frac{1}{12}$$

$$= \frac{3}{12} - \frac{1}{12}$$

$$= \frac{2}{12}$$

$$= \frac{1}{6}$$

i.e. true for $n=1$.

Assume true for $n=k$: i.e. $\displaystyle\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$, $k \geq 1$, $k \in \mathbb{N}$ --- (1)

Want to prove: $\displaystyle\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1+1)(k+1+2)}$

$$= \frac{1}{4} - \frac{1}{2(k+2)(k+3)} \quad , k \geq 1, k \in \mathbb{N} \ldots (2)$$

$n=k+1$: $\displaystyle\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \qquad [\text{using } (1)]$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{2}{2(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)}\left[1 - \frac{2}{(k+3)}\right]$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)}\left[\frac{k+3}{k+3} - \frac{2}{k+3}\right]$$

$$= \frac{1}{4} - \frac{1}{2\cancel{(k+1)}(k+2)}\left[\frac{\cancel{(k+1)}}{(k+3)}\right] = \frac{1}{4} - \frac{1}{2(k+2)(k+3)} \text{ --- (2) as required.}$$

i.e. true for $n=k+1$ (etc).

(6) Prove: n^3+3n^2-10n is divisible by 3, $n \in \mathbb{N}$

i.e. prove: $n^3+3n^2-10n = 3p$, $p \in \mathbb{Z}$, $n \in \mathbb{N}$.

$\underline{n=1}$ n^3+3n^2-10n

$= 1^3+3\times1^2-10\times1$

$= 1+3-10$

$= -6$

$= 3\times-2$ i.e. $3p$ where $p=-2$, so $p \in \mathbb{Z}$.

i.e. n^3+3n^2-10n is divisible by 3 for $n=1$ i.e. true for $n=1$.

Assume true for $n=k$: i.e. $k^3+3k^2-10k = 3q$, $q \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (1)

Want to prove: $(k+1)^3+3(k+1)^2-10(k+1) = 3t$, $t \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (2)

$\underline{n=k+1}$: $(k+1)^3+3(k+1)^2-10(k+1)$

$= k^3+3k^2+3k+1+3(k^2+2k+1)-10k-10$

$= k^3+3k^2+3k+1+3k^2+6k+3-10k-10$

$= k^3+3k^2-10k+3k^2+3k+6k+1+3-10$ (i.e. same terms, different order).

$= 3q + 3k^2+9k-6$ ---- [using (1)]

$= 3(q+k^2+3k-2)$

$= 3t$, $t = q+k^2+3k-2$, so $t \in \mathbb{Z}$.

which is (2), as required.

i.e. true for $n=k+1$.

(etc)

(6) Prove: n^3+3n^2-10n is divisible by 3, $n \in \mathbb{N}$

i.e. prove: $n^3+3n^2-10n = 3p$, $p \in \mathbb{Z}$, $n \in \mathbb{N}$.

$\underline{n=1}$ n^3+3n^2-10n

$= 1^3+3\times1^2-10\times1$

$= 1+3-10$

$= -6$

$= 3\times-2$ i.e. $3p$ where $p=-2$, so $p \in \mathbb{Z}$.

i.e. n^3+3n^2-10n is divisible by 3 for $n=1$ i.e. true for $n=1$.

Assume true for $n=k$: i.e. $k^3+3k^2-10k = 3q$, $q \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (1)

Want to prove: $(k+1)^3+3(k+1)^2-10(k+1) = 3t$, $t \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (2)

$\underline{n=k+1}$: $(k+1)^3+3(k+1)^2-10(k+1)$

$= k^3+3k^2+3k+1+3(k^2+2k+1)-10k-10$

$= k^3+3k^2+3k+1+3k^2+6k+3-10k-10$

$= k^3+3k^2-10k+3k^2+3k+6k+1+3-10$ (i.e. same terms, different order).

$= 3q + 3k^2+9k-6$ ---- [using (1)]

$= 3(q+k^2+3k-2)$

$= 3t$, $t = q+k^2+3k-2$, so $t \in \mathbb{Z}$.

which is (2), as required.

i.e. true for $n=k+1$.

(etc)

⑦ Prove: $7^n+4^n+1^n$ is divisible by 6, $n\in\mathbb{N}$.

i.e. prove $7^n+4^n+1^n = 6p$, $p\in\mathbb{Z}$, $n\in\mathbb{N}$.

$\underline{n=1}$: $7^1+4^1+1^1 = 7+4+1 = 12 = 6\times 2$.

i.e. it is divisible by 6.

true for $n=1$.

Assume true for $n=k$: i.e. $7^k+4^k+1^k = 6q$, $q\in\mathbb{Z}$, $k\in\mathbb{N}$ --- (1)

Want to prove: $7^{k+1}+4^{k+1}+1^{k+1} = 6t$, $t\in\mathbb{Z}$, $k\in\mathbb{N}$ --- (2).

$\underline{n=k+1}$: $7^{k+1}+4^{k+1}+1^{k+1}$

$= 7.7^k+4.4^k+1$ (since 1 to any power is 1).

$= 7(7^k+4^k+1^k)-7.4^k-7.1^k+4.4^k+1$

$= 7(6q)-3.4^k-7+1$ (since $1^k=1$) [using (1) at start].

$= 42q-3.4^k-6 = 42q-3\times 4\times 4^{k-1}-6 = 42q-12\times 4^{k-1}-6$

$= 6(7q-2.4^{k-1}-1)$

$= 6t$, where $t=7q-2.4^{k-1}-1$, so $t\in\mathbb{Z}$

which is (2), as required.

i.e. true for $n=k+1$.

⑧ Prove: $2^n > 2n$, $n>2$, $n\in\mathbb{N}$.

i.e. prove $2^n > 2n$, $n\geqslant 3$, $n\in\mathbb{N}$.

$\underline{n=3}$ $\underline{\text{LHS}}$ $2^3=8$ $\underline{\text{RHS}}$ $2\times 3=6$.

$8>6$.

i.e. true for $n=3$.

Assume true for $n=k$: i.e. assume $2^k > 2k$, $k\geqslant 3$, $k\in\mathbb{N}$ --- (1)

Want to prove: $2^{k+1} > 2(k+1)$, $k\geqslant 3$, $k\in\mathbb{N}$ --- (2).

$\underline{n=k+1}$: $2^{k+1} = 2.2^k > 2.2k$ [using (1)]

i.e. $2^{k+1} > 4k$

i.e. $2^{k+1} > 2k+2+2k-2$

$\therefore$ $2^{k+1} > 2(k+1)+2k-2 > 2(k+1)$ since $2k-2\geqslant 4>0$ $(k\geqslant 3)$

i.e. $2^{k+1} > 2(k+1)$, $k\geqslant 3$ which is (2), as required. (etc).

i.e. true for $n=k+1$.

(7) Prove: $7^n + 4^n + 1^n$ is divisible by 6, $n \in \mathbb{N}$.

i.e. prove $7^n + 4^n + 1^n = 6p$, $p \in \mathbb{Z}$, $n \in \mathbb{N}$.

n=1: $7^1 + 4^1 + 1^1 = 7 + 4 + 1 = 12 = 6 \times 2$.

i.e. it is divisible by 6.

true for $n=1$.

Assume true for $n=k$: i.e. $7^k + 4^k + 1^k = 6q$, $q \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (1)

Want to prove: $7^{k+1} + 4^{k+1} + 1^{k+1} = 6t$, $t \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (2).

n=k+1: $7^{k+1} + 4^{k+1} + 1^{k+1}$

$= 7.7^k + 4.4^k + 1$ (since 1 to any power is 1)

$= 7(7^k + 4^k + 1^k) - 7.4^k - 7.1^k + 4.4^k + 1$

$= 7(6q) - 3.4^k - 7 + 1$ (since $1^k = 1$) [using (1) at start].

$= 42q - 3.4^k - 6 = 42q - 3 \times 4 \times 4^{k-1} - 6 = 42q - 12 \times 4^{k-1} - 6$

$= 6(7q - 2.4^{k-1} - 1)$

$= 6t$, where $t = 7q - 2.4^{k-1} - 1$, so $t \in \mathbb{Z}$

which is (2), as required.

i.e. true for $n = k+1$.

(8) Prove: $2^n > 2n$, $n > 2$, $n \in \mathbb{N}$.

i.e. prove $2^n > 2n$, $n \geq 3$, $n \in \mathbb{N}$.

n=3 LHS $2^3 = 8$ RHS $2 \times 3 = 6$.

$8 > 6$.

i.e. true for $n=3$.

Assume true for $n=k$: i.e. assume $2^k > 2k$, $k \geq 3$, $k \in \mathbb{N}$ --- (1)

Want to prove: $2^{k+1} > 2(k+1)$, $k \geq 3$, $k \in \mathbb{N}$ --- (2).

n=k+1: $2^{k+1} = 2.2^k > 2.2k$ [using (1)]

i.e. $2^{k+1} > 4k$

i.e. $2^{k+1} > 2k + 2 + 2k - 2$

$\therefore 2^{k+1} > 2(k+1) + 2k - 2 > 2(k+1)$ since $2k - 2 \geq 4 > 0$ $(k \geq 3)$

i.e. $2^{k+1} > 2(k+1)$, $k \geq 3$ which is (2), as required. (etc).

i.e. true for $n=k+1$.

(9) Prove: $3^n > n^3$ for all $n \geq 4$, $n \in \mathbb{N}$.

$n=4$ LHS $3^4 = 81$ RHS $4^3 = 64$.

$81 > 64$ so true for $n=4$.

Assume true for $n=k$: ie. $3^k > k^3$ for all $k \geq 4$, $k \in \mathbb{N}$ ---- (1).

Want to prove: $3^{k+1} > (k+1)^3$, $k \geq 4$, $k \in \mathbb{N}$ ---- (2).

$n=k+1$: $3^{k+1} = 3 \cdot 3^k > 3 \cdot k^3$ [using (1)]

i.e. $3^{k+1} > k^3 + 2k^3$

i.e. $3^{k+1} > (k+1)^3 - 3k^2 - 3k - 1 + 2k^3$

ie. $3^{k+1} > (k+1)^3 + 2k^3 - 3k^2 - 3k - 1 > (k+1)^3$ ---- (*)

provided that $2k^3 - 3k^2 - 3k - 1 > 0$ for all $k \geq 4$, $k \in \mathbb{N}$ ---- (3)

If (3) is true then (dividing by k^3):

$$2 - \frac{3}{k} - \frac{3}{k^2} - \frac{1}{k^3} > 0 \text{ must be true} \text{ ---- (4).}$$

But (4) is true when $k=4$ since $2 - \frac{3}{4} - \frac{3}{4^2} - \frac{1}{4^3} = 2 - \frac{3}{4} - \frac{3}{16} - \frac{1}{64}$

$= 1\frac{3}{64}$ which is > 0

and when $k \to \infty$, $\frac{3}{k} \to 0$, $\frac{3}{k^2} \to 0$ and $\frac{1}{k^3} \to 0$.

so $2 - \frac{3}{k} - \frac{3}{k^2} - \frac{1}{k^3} \to 2$ as $k \to \infty$

ie. $2 - \frac{3}{k} - \frac{3}{k^2} - \frac{1}{k^3} > 0$ for all $k \geq 4$, $k \in \mathbb{N}$

so (4) is true

$\Rightarrow$ (3) is true

$\Rightarrow$ (*) is true.

i.e. $3^{k+1} > (k+1)^3$, $k \geq 4$, $k \in \mathbb{N}$ which is (2), as required.

ie. true for $n = k+1$

(etc).

(7) Prove: $7^n + 4^n + 1^n$ is divisible by 6, $n \in \mathbb{N}$.

i.e. prove $7^n + 4^n + 1^n = 6p$, $p \in \mathbb{Z}$, $n \in \mathbb{N}$.

<u>n=1:</u> $7^1 + 4^1 + 1^1 = 7 + 4 + 1 = 12 = 6 \times 2$.

i.e. it is divisible by 6.

true for n=1.

Assume true for n=k: i.e. $7^k + 4^k + 1^k = 6q$, $q \in \mathbb{Z}$, $k \in \mathbb{N}$ --(1)

Want to prove: $7^{k+1} + 4^{k+1} + 1^{k+1} = 6t$, $t \in \mathbb{Z}$, $k \in \mathbb{N}$ --(2).

<u>n=k+1:</u> $7^{k+1} + 4^{k+1} + 1^{k+1}$

$= 7.7^k + 4.4^k + 1$ (since 1 to any power is 1)

$= 7(7^k + 4^k + 1^k) - 7.4^k - 7.1^k + 4.4^k + 1$

$= 7(6q) - 3.4^k - 7 + 1$ (since $1^k = 1$) [using (1) at start].

$= 42q - 3.4^k - 6 = 42q - 3 \times 4 \times 4^{k-1} - 6 = 42q - 12 \times 4^{k-1} - 6$

$= 6(7q - 2.4^{k-1} - 1)$

$= 6t$, where $t = 7q - 2.4^{k-1} - 1$, so $t \in \mathbb{Z}$

which is (2), as required.

i.e. true for n=k+1.

(8) Prove: $2^n > 2n$, $n > 2$, $n \in \mathbb{N}$.

i.e. prove $2^n > 2n$, $n \geq 3$, $n \in \mathbb{N}$.

<u>n=3</u> <u>LHS</u> $2^3 = 8$ <u>RHS</u> $2 \times 3 = 6$.

$8 > 6$.

i.e. true for n=3.

Assume true for n=k: i.e. assume $2^k > 2k$, $k \geq 3$, $k \in \mathbb{N}$ --(1)

Want to prove: $2^{k+1} > 2(k+1)$, $k \geq 3$, $k \in \mathbb{N}$ --------(2).

<u>n=k+1:</u> $2^{k+1} = 2.2^k > 2.2k$ [using (1)]

i.e. $2^{k+1} > 4k$

i.e. $2^{k+1} > 2k + 2 + 2k - 2$

$\therefore 2^{k+1} > 2(k+1) + 2k - 2 > 2(k+1)$ since $2k - 2 \geq 4 > 0$ $(k \geq 3)$

ie $2^{k+1} > 2(k+1)$, $k \geq 3$ which is (2), as required. (etc).

i.e. true for n=k+1.

(6) Prove: n^3+3n^2-10n is divisible by 3, $n \in \mathbb{N}$

i.e. prove: $n^3+3n^2-10n = 3p$, $p \in \mathbb{Z}$, $n \in \mathbb{N}$.

$\underline{n=1}$ $\quad n^3+3n^2-10n$

$= 1^3+3\times1^2-10\times1$

$= 1+3-10$

$= -6$

$= 3\times-2$ i.e. $3p$ where $p=-2$, so $p \in \mathbb{Z}$.

i.e. n^3+3n^2-10n is divisible by 3 for $n=1$ i.e. true for $n=1$.

Assume true for $n=k$: i.e. $k^3+3k^2-10k = 3q$, $q \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (1)

Want to prove: $(k+1)^3+3(k+1)^2-10(k+1) = 3t$, $t \in \mathbb{Z}$, $k \in \mathbb{N}$ --- (2)

$\underline{n=k+1}$: $(k+1)^3+3(k+1)^2-10(k+1)$

$= k^3+3k^2+3k+1+3(k^2+2k+1)-10k-10$

$= k^3+3k^2+3k+1+3k^2+6k+3-10k-10$

$= k^3+3k^2-10k+3k^2+3k+6k+1+3-10$ (i.e. same terms, different order).

$= 3q + 3k^2+9k-6$ ---- [using (1)]

$= 3(q+k^2+3k-2)$

$= 3t$, $t = q+k^2+3k-2$, so $t \in \mathbb{Z}$.

which is (2), as required.

i.e. true for $n=k+1$.

(etc)

④ S_n of the series $\underbrace{\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\dots}_{n \text{ terms}}$ can be written as:

$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$$

i.e. Prove $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$, $n\geq1,\ n\in\mathbb{N}$ --- (1).

$n=1$ LHS $\sum_{r=1}^{1}\frac{1}{r(r+1)(r+2)}$

$=\frac{1}{1\times2\times3}$

$=\frac{1}{6}$

RHS $\frac{1}{4}-\frac{1}{2(1+1)(1+2)}$

$=\frac{1}{4}-\frac{1}{2\times2\times3}$

$=\frac{1}{4}-\frac{1}{12}$

$=\frac{3}{12}-\frac{1}{12}$

$=\frac{2}{12}$

$=\frac{1}{6}$

i.e. true for $n=1$.

Assume true for $n=k$: i.e. $\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}$, $k\geq1,\ k\in\mathbb{N}$ --- (1)

Want to prove: $\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(k+1+1)(k+1+2)}$

$=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}$, $k\geq1,\ k\in\mathbb{N}$ ... (2)

$n=k+1$: $\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$ [using (1)]

$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{2}{2(k+1)(k+2)(k+3)}$

$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}\left[1-\frac{2}{(k+3)}\right]$

$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}\left[\frac{k+3}{k+3}-\frac{2}{k+3}\right]$

$=\frac{1}{4}-\frac{1}{2\cancel{(k+1)}(k+2)}\left[\frac{\cancel{(k+1)}}{(k+3)}\right]=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}$ --- (2) as required.

i.e. true for $n=k+1$ (etc).

③ S_n of $\underbrace{\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots}_{n \text{ terms}}$

can be written as $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$

i.e. Prove $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$, $n \geq 1$, $n \in \mathbb{N}$

$\underline{n=1}$ LHS $\displaystyle\sum_{r=1}^{1} \frac{1}{(2r-1)(2r+1)}$ RHS $\dfrac{n}{2n+1}$

$$= \frac{1}{(2\times 1-1)(2\times 1+1)} \qquad = \frac{1}{2\times 1+1}$$

$$= \frac{1}{1\times 3} \qquad = \frac{1}{2+1}$$

$$= \frac{1}{3} \qquad = \frac{1}{3}$$

i.e. true for $n=1$.

Assume true for $n=k$: i.e. $\displaystyle\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$, $k \geq 1$, $k \in \mathbb{N}$ ---- (1)

Want to prove: $\displaystyle\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$, $k \geq 1$, $k \in \mathbb{N}$ ---- (2)

$\underline{n=k+1}$: $\displaystyle\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \qquad \text{---------- using (1)}$$

$$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$ which is (2), as required. i.e. true for $n=k+1$ (etc)

(2) Prove $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$, $n \geq 1$, $n \in \mathbb{N}$.

$n=1$: LHS $\sum_{r=1}^{1} r(r+1)(r+2)$ RHS $\frac{1}{4}(1)(1+1)(1+2)(1+3)$

$= 1(1+1)(1+2)$ $= \frac{1}{4} \times 1 \times 2 \times 3 \times 4$

$= 1 \times 2 \times 3$ $= \frac{1}{4} \times$

$= 6$ $= 6$

$\therefore$ True for $n=1$.

Assume true for $n=k$: i.e. $\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$, $k \geq 1$, $k \in \mathbb{N}$. ---- (1)

Want to prove : $\sum_{r=1}^{k+1} r(r+1)(r+2) = \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ ----- (2)

$n=k+1$: $\sum_{r=1}^{k+1} r(r+1)(r+2) = \sum_{r=1}^{k} r(r+1)(r+2) + (k+1)(k+2)(k+3)$

$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$ ...using (1)

$= \frac{1}{4}(k+1)(k+2)(k+3)\left[k+4\right]$ --- $\frac{1}{4}(k+1)(k+2)(k+3)$ as common factor

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ which is (2) as req^{d}.

i.e. true for $n=k+1$.

[then usual summary]

TJ3. p.85 Ex5 Nos 1→4, 6→9 Proof by Induction

① Prove $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$, $n \geq 1$, $n \in \mathbb{N}$.

$n=1$: LHS $\sum_{r=1}^{1} r(r+1)$ RHS $\frac{1}{3}(1)(1+1)(1+2)$

$= 1(1+1)$ $= \frac{1}{3} \times 1 \times 2 \times 3$

$= 1 \times 2$ $= \frac{1}{3} \times 6$

$= 2$ $= 2$

∴ True for $n=1$.

Assume true for $n=k$: $\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$ ------ (1)

Want to prove: $\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)(k+3)$ ------ (2)

$n=k+1$:

$$\sum_{r=1}^{k+1} r(r+1) = \sum_{r=1}^{k} r(r+1) + (k+1)(k+2)$$

$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$ --- using (1)

$= \frac{1}{3}(k+1)(k+2)\left[k+3\right]$ ---- (taking $\frac{1}{3}(k+1)(k+2)$ as a common factor).

$= \frac{1}{3}(k+1)(k+2)(k+3)$ which is (2), as required.

i.e. true for $n=k+1$.

Summary True for $n=1$ and truth of $n=k \Rightarrow$ true for $n=k+1$
ie. true for $n=1 \Rightarrow$ true for $n=2$
true for $n=2 \Rightarrow$ true for $n=3$
etc.
ie. true for $n \geq 1$, $n \in \mathbb{N}$. *

[* This kind of summary at the end will be omitted for the remaining solutions, but should really be written each time.]