Answers to "Proof by Induction (Unit 2)"

Q1 (i) Prove that $\forall n \in \mathbb{N}$, $\frac{1}{1\times2}+\frac{1}{2\times3}+\dots+\frac{1}{n(n+1)} = \frac{n}{n+1}$.

Ans 1(i) Prove that $\forall n \in \mathbb{N}$, $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

$n=1$ LHS $\sum_{r=1}^{1} \frac{1}{r(r+1)}$ $= \frac{1}{1(1+1)}$ $= \frac{1}{1\times2}$ $= \frac{1}{2}$

RHS $\frac{1}{1+1}$ $= \frac{1}{2}$.

LHS = RHS
so true for $n=1$.

Assume true for $n=k$.
i.e. assume that $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$ ------- (1)

$n=k+1$: Required to prove: $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{(k+1)+1}$

i.e. $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$ ------- (2)

$n=k+1$ LHS. $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + (k+1)^{th}$ term

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+1+1)} \quad \text{[using (1)]}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{k+1}\left[k + \frac{1}{k+2}\right]$$

$$= \frac{1}{k+1}\left[\frac{k(k+2)}{k+2} + \frac{1}{k+2}\right] = \frac{1}{k+1}\left[\frac{k^2+2k+1}{k+2}\right]$$

$$= \frac{1}{\cancel{(k+1)}}\left[\frac{(k+1)^{\cancel{2}}}{k+2}\right] = \frac{k+1}{k+2}$$ which is (2) as required.
i.e. true for $n=k+1$.

Conclusion It is true for $n=1 \Rightarrow$ true for $n=2 \Rightarrow$ true for $n=3$, and so on, since assuming true for $n=k \Rightarrow$ true for $n=k+1$. $\Rightarrow$ original statement true $\forall n \in \mathbb{N}$.

Q1 (ii) Prove $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$, $\forall n \in \mathbb{N}$.

Ans 1 (ii) <u>n=1</u> LHS $\sum_{r=1}^{1} r(r+1)(r+2)$ RHS $\frac{1}{4}(1)(1+1)(1+2)(1+3)$

$= 1(1+1)(1+2)$
$= 1\times 2\times 3$
$= 6$

$= \frac{1}{4}\times 1\times 2\times 3\times 4$
$= \frac{1}{4}\times 24$
$= 6.$

LHS = RHS so true for n=1.

<u>Assume true for n=k</u> i.e. assume that $\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$ ......(1)

n=k+1: Required to prove: $\sum_{r=1}^{k+1} r(r+1)(r+2) = \frac{1}{4}(k+1)(k+1+1)(k+1+2)(k+1+3)$

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ ....(2)

<u>n=k+1</u> LHS $\sum_{r=1}^{k+1} r(r+1)(r+2) = \sum_{r=1}^{k} r(r+1)(r+2) + (k+1)^{th}$ term

$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+1+1)(k+1+2)$
[using (1)]

$= \frac{1}{4}k(k+1)(k+2)(k+3) + \frac{4}{4}(k+1)(k+2)(k+3)$

$= \frac{1}{4}(k+1)(k+2)(k+3)\left[k+4\right]$

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$ which is (2) as required.

i.e. true for n=k+1.

<u>Conclusion:</u> True for n=1 ⇒ true for n=2 ⇒ **true for n=3**, and so on, since assuming true for n=k ⇒ true for n=k+1

⇒ original statement is true $\forall n \in \mathbb{N}$.

Q 1 (iii) Prove $\sum_{r=0}^{n-1} ax^r = \frac{a(x^n-1)}{x-1}$, $x \neq 1$, $\forall n \in \mathbb{N}$

Ans 1 (iii) n=1 LHS $\sum_{r=0}^{1-1} ax^r$ RHS $\frac{a(x^1-1)}{x-1}$

$= \sum_{r=0}^{0} ax^r$ $= \frac{a\cancel{(x-1)}}{\cancel{(x-1)}}$

$= ax^0$

$= a \times 1$ $= a$

$= a$

LHS = RHS so true for n=1.

Assume true for n=k

i.e. assume that $\sum_{r=0}^{k-1} ax^r = \frac{a(x^k-1)}{x-1}$ $(x \neq 1)$ ------(1)

n=k+1: Required to prove that $\sum_{r=0}^{k} ax^r = \frac{a(x^{k+1}-1)}{x-1}$ $(x \neq 1)$ ------(2)

n=k+1 LHS $\sum_{r=0}^{k} ax^r = \sum_{r=0}^{k-1} ax^r + (k+1)^{th}$ term

$= \sum_{r=0}^{k-1} ax^r + ax^k$

1st term : r=0
2nd term : r=1
3rd term : r=2
⋮
k^{th} term : r=k-1
$(k+1)^{th}$ term : r=k.

$= \frac{a(x^k-1)}{x-1} + ax^k$ [using (1)]

$= \frac{a(x^k-1)}{x-1} + \frac{ax^k(x-1)}{x-1}$

$= \frac{ax^k - a + ax^{k+1} - ax^k}{x-1}$

$= \frac{ax^{k+1}-a}{x-1}$

$= \frac{a(x^{k+1}-1)}{x-1}$ which is (2), as required

Conclusion It's true for n=1 ⇒ true for n=2 ⇒ true for n=3, etc, since assuming true for n=k ⇒ true for n=k+1 ⇒ original statement true $\forall n \in \mathbb{N}$.

Q1(iv) Prove that $\sum_{r=1}^{n} (-1)^{r-1} r^2 = \frac{1}{2}(-1)^{n-1} n(n+1)$

Ans 1(iv) $\underline{n=1}$ LHS $\sum_{r=1}^{1} (-1)^{r-1} r^2$

$$= (-1)^{1-1} 1^2 = (-1)^0 \times 1 = 1 \times 1 = 1$$

RHS $\frac{1}{2}(-1)^{1-1}(1)(1+1)$

$$= \frac{1}{2}(-1)^0(1)(2) = \frac{1}{2} \times 1 \times 1 \times 2 = \frac{1}{2} \times 2 = 1$$

LHS = RHS, so true for $n=1$.

Assume true for $n=k$.

i.e. assume that $\sum_{r=1}^{k} (-1)^{r-1} r^2 = \frac{1}{2}(-1)^{k-1} k(k+1)$ ....(1)

$n=k+1$: Required to prove that $\sum_{r=1}^{k+1} (-1)^{r-1} r^2 = \frac{1}{2}(-1)^{k}(k+1)(k+2)$.......(2)

$\underline{n=k+1}$ LHS $\sum_{r=1}^{k+1} (-1)^{r-1} r^2 = \sum_{r=1}^{k} (-1)^{r-1} r^2 + (k+1)^{th}$ term

$$= \sum_{r=1}^{k} (-1)^{r-1} r^2 + (-1)^{k+1-1}(k+1)^2$$

$$= \frac{1}{2}(-1)^{k-1} k(k+1) + (-1)^k (k+1)^2 \quad \text{[using (1)]}$$

$$= \frac{1}{2}(-1)^{k-1}(k+1)\left[k + 2(-1)(k+1)\right]$$

$$= \frac{1}{2}(-1)^{k-1}(k+1)\left[k - 2k - 2\right]$$

$$= \frac{1}{2}(-1)^{k-1}(k+1)(-k-2)$$

$$= \frac{1}{2}(-1)^{k-1}(k+1)(-1)(k+2)$$

$$= \frac{1}{2}(-1)^{k}(k+1)(k+2)$$ which is (2), as required.

Conclusion It's true for $n=1 \Rightarrow$ true for $n=2 \Rightarrow$ true for $n=3$, etc, since assuming true for $n=k \Rightarrow$ true for $n=k+1 \Rightarrow$ original statement true $\forall n \in \mathbb{N}$.

Q1(v) Prove that $\sum_{r=1}^{n} r^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2+2n-1) \quad \forall n \in \mathbb{N}$.

Ans 1(v) $\underline{n=1}$ LHS $\sum_{r=1}^{1} r^5$ RHS $\frac{1}{12}(1)^2(1+1)^2(2(1)^2+2(1)-1)$

$= 1^5$

$= 1$

$= \frac{1}{12} \times 1^2 \times 2^2 \times 3$

$= \frac{1}{12} \times 1 \times 4 \times 3$

$= \frac{1}{12} \times 12$

$= 1$

LHS = RHS, so true for $n=1$.

Assume true for $n=k$

i.e. assume that $\sum_{r=1}^{k} r^5 = \frac{1}{12} k^2 (k+1)^2 (2k^2+2k-1)$ ------(1)

$n=k+1$: Required to prove that $\sum_{r=1}^{k+1} r^5 = \frac{1}{12}(k+1)^2(k+2)^2\left[2(k+1)^2+2(k+1)-1\right]$

$= \frac{1}{12}(k+1)^2(k+2)^2\left[2(k^2+2k+1)+2k+2-1\right]$

$= \frac{1}{12}(k+1)^2(k+2)^2\left[2k^2+4k+2+2k+1\right]$

$= \frac{1}{12}(k+1)^2(k+2)^2(2k^2+6k+3)$ ----(2)

$\underline{n=k+1}$ $\underline{\text{LHS}}$ $\sum_{r=1}^{k+1} r^5 = \sum_{r=1}^{k} r^5 + (k+1)^{th}$ term

$= \sum_{r=1}^{k} r^5 + (k+1)^5$

$= \frac{1}{12} k^2 (k+1)^2 (2k^2+2k-1) + (k+1)^5$ [using (1)]

$= \frac{1}{12}(k+1)^2\left[k^2(2k^2+2k-1)+12(k+1)^3\right]$

$= \frac{1}{12}(k+1)^2\left[2k^4+2k^3-k^2+12(k^3+3k^2+3k+1)\right]$

$= \frac{1}{12}(k+1)^2\left[2k^4+2k^3-k^2+12k^3+36k^2+36k+12\right]$

(v) (cont^d)

$$= \frac{1}{12}(k+1)^2\left[2k^4+14k^3+35k^2+36k+12\right]$$

[Aside:

$$\begin{array}{r|rrrrr} -2 & 2 & 14 & 35 & 36 & 12 \\ & & -4 & -20 & -30 & -12 \\ \hline & 2 & 10 & 15 & 6 & 0 \end{array}$$

$(k+2)(2k^3+10k^2+15k+6)$

$$\begin{array}{r|rrrr} -2 & 2 & 10 & 15 & 6 \\ & & -4 & -12 & -6 \\ \hline & 2 & 6 & 3 & 0 \end{array}$$

$(k+2)(k+2)(2k^2+6k+3)$

$(k+2)^2(2k^2+6k+3)$]

so $\frac{1}{12}(k+1)^2\left[2k^4+14k^3+35k^2+36k+12\right]$

$= \frac{1}{12}(k+1)^2(k+2)^2(2k^2+6k+3)$ which is (2), as required

Conclusion It's true for $n=1 \Rightarrow$ true for $n=2 \Rightarrow$ true for $n=3$, etc, since assuming true for $n=k \Rightarrow$ true for $n=k+1 \Rightarrow$ original statement true $\forall n \in \mathbb{N}$.

Q1 (vi) Prove that $1(2)^2+2(3)^2+\dots+n(n+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5)$

Ans 1 (vi) Prove $\sum_{r=1}^{n} r(r+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5)$

$n=1$ LHS $\sum_{r=1}^{1} r(r+1)^2 = 1(1+1)^2 = 1\times 2^2 = 4.$

RHS $\frac{1}{12}(1)(1+1)(1+2)(3(1)+5)$

$= \frac{1}{12}\times 1\times 2\times 3\times 8 = \frac{1}{12}\times 48 = 4$

LHS = RHS, so true for $n=1$.

Assume true for $n=k$: i.e. assume $\sum_{r=1}^{k} r(r+1)^2 = \frac{1}{12}k(k+1)(k+2)(3k+5)$ ---(1)

$n=k+1$: Required to prove $\sum_{r=1}^{k+1} r(r+1)^2 = \frac{1}{12}(k+1)(k+1+1)(k+1+2)(3(k+1)+5)$

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+3+5)$

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$ ---(2)

(vi)(cont^d)

$\underline{n=k+1}$: LHS $1\times 2^2+2\times 3^2+\ldots+k(k+1)^2+(k+1)(k+2)^2$

$$= \frac{1}{12}k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2 \qquad \text{(using } * \text{)}$$

$$= \frac{1}{12}(k+1)(k+2)\left[k(3k+5) + 12(k+2)\right]$$

$$= \frac{1}{12}(k+1)(k+2)\left[3k^2+5k+12k+24\right]$$

$$= \frac{1}{12}(k+1)(k+2)(3k^2+17k+24)$$

$$= \frac{1}{12}(k+1)(k+2)(3k+8)(k+3)$$ — RHS of what we were required to prove.

$\therefore$ True for $n=1$

Given true for $n=k \Rightarrow$ true for $n=k+1$.

$\therefore$ Proof by induction.

(vii) Prove that $\forall\, n\in\mathbb{N}$, $\displaystyle\sum_{r=1}^{n}\frac{(r+1)^2}{r(r+2)} = \frac{n(4n^2+15n+13)}{4(n+1)(n+2)}$

$\underline{n=1}$ LHS $\displaystyle\sum_{r=1}^{1}\frac{(r+1)^2}{r(r+2)} = \frac{(1+1)^2}{1(1+2)} = \frac{4}{3}$

RHS $\displaystyle\frac{1(4(1)^2+15(1)+13)}{4(1+1)(1+2)} = \frac{32}{24} = \frac{4}{3}$ $\therefore$ True for $n=1$.

Assume true for $n=k$:

ie. assume that $\displaystyle\sum_{r=1}^{k}\frac{(r+1)^2}{r(r+2)} = \frac{k(4k^2+15k+13)}{4(k+1)(k+2)}$ - - - $(*)$

Required to prove: $\displaystyle\sum_{r=1}^{k+1}\frac{(r+1)^2}{r(r+2)} = \frac{(k+1)\left[4(k+1)^2+15(k+1)+13\right]}{4(k+2)(k+3)}$

$$= \frac{(k+1)\left[4k^2+8k+4+15k+15+13\right]}{4(k+2)(k+3)}$$

$$= \frac{(k+1)(4k^2+23k+32)}{4(k+2)(k+3)}$$

(vii) (cont^d)

$\underline{n=k+1}$ LHS

$$\sum_{r=1}^{k+1} \frac{(r+1)^2}{r(r+2)} = \sum_{r=1}^{k} \frac{(r+1)^2}{r(r+2)} + \frac{(k+2)^2}{(k+1)(k+3)}$$

$$= \frac{k(4k^2+15k+13)}{4(k+1)(k+2)} + \frac{(k+2)^2}{(k+1)(k+3)}$$

$$= \frac{k(4k^2+15k+13)(k+3)+4(k+2)^2(k+2)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(4k^3+15k^2+13k)(k+3)+4(k+2)^3}{4(k+1)(k+2)(k+3)}$$

$$= \frac{4k^4+15k^3+13k^2+12k^3+45k^2+39k+4(k^3+3k^2(2)+3k(2)^2+2^3)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{4k^4+27k^3+58k^2+39k+4k^3+24k^2+48k+32}{4(k+1)(k+2)(k+3)}$$

$$= \frac{4k^4+31k^3+82k^2+87k+32}{4(k+1)(k+2)(k+3)}$$

[Aside:
$$\begin{array}{r|rrrrr} -1 & 4 & 31 & 82 & 87 & 32 \\ & & -4 & -27 & -55 & -32 \\ \hline & \boxed{4 \quad 27} & 55 & 32 & & 0 \end{array}$$
]

$$= \frac{\cancel{(k+1)}(4k^3+27k^2+55k+32)}{4\cancel{(k+1)}(k+2)(k+3)}$$

$$= \frac{4k^3+27k^2+55k+32}{4(k+2)(k+3)}$$

[Aside:
$$\begin{array}{r|rrrr} -1 & 4 & 27 & 55 & 32 \\ & & -4 & -23 & -32 \\ \hline & 4 & 23 & 32 & 0 \end{array}$$
]

$$= \frac{(k+1)(4k^2+23k+32)}{4(k+2)(k+3}$$

— RHS of what we were required to prove.

$\therefore$ ...... Proof by induction.

(viii) Prove that $\forall\, n \in \mathbb{N},\ \sum_{r=1}^{2n} (-1)^r r^3 = n^2(4n+3)$

$\underline{n=1}$ LHS $\sum_{r=1}^{2} (-1)^r r^3 = (-1)^1 1^3 + (-1)^2 2^3 = -1 + 8 = 7$

RHS $1^2(4\times 1+3) = 1\times 7 = 7$ $\quad\therefore$ True for $n=1$.

Assume true for $n=k$

ie. assume $\sum_{r=1}^{2k} (-1)^r r^3 = k^2(4k+3)$ - - - - - - - (*)

Required to prove that $\sum_{r=1}^{2(k+1)} (-1)^r r^3 = (k+1)^2\big(4(k+1)+3\big)$

ie. that $\sum_{r=1}^{2k+2} (-1)^r r^3 = (k+1)^2(4k+7)$

$\underline{n=k+1}$ LHS $\sum_{r=1}^{2k+2} (-1)^r r^3 = \sum_{r=1}^{2k} (-1)^r r^3 + (-1)^{2k+1}(2k+1)^3 + (-1)^{2k+2}(2k+2)^3$

$$
\begin{aligned}
&= k^2(4k+3) + (-1)^{2k+1}\left[(2k+1)^3 - (2k+2)^3\right]\\
&= k^2(4k+3) + (-1)^{2k+1}\left[(2k)^3 + 3(2k)^2 + 3(2k) + 1 - 8(k+1)^3\right]\\
&= k^2(4k+3) + (-1)^{2k+1}\left[\cancel{8k^3} + 12k^2 + 6k + 1 \cancel{- 8k^3} - 24k^2 - 24k - 8\right]\\
&= k^2(4k+3) + (-1)^{2k+1}\left[-12k^2 - 18k - 7\right]\\
&= k^2(4k+3) + (-1)^{2k+2}(12k^2 + 18k + 7)\\
&= k^2(4k+3) + (-1)^{2(k+1)}(12k^2 + 18k + 7)\\
&= k^2(4k+3) + (12k^2 + 18k + 7) \quad \left[\text{Since } (-1)^{\text{even}} = 1\right]\\
&= 4k^3 + 3k^2 + 12k^2 + 18k + 7\\
&= 4k^3 + 15k^2 + 18k + 7\\
&= (k+1)(4k^2 + 11k + 7)\\
&= (k+1)(k+1)(4k+7)\\
&= (k+1)^2(4k+7) \longrightarrow \left[\text{RHS of what we were required to prove.}\right]
\end{aligned}
$$

$$
\left[\begin{array}{c|c|c} 12k & 7 & 1 \\ \hline k & 1 & 7 \\ \\ 6k & 7 & 1 \\ \hline 2k & 1 & 7 \\ \\ 4k & 7 & 1 \\ \hline 3k & 1 & 7 \end{array}\right\}\ \text{Doesn't factorise}
$$

$$
\begin{array}{r|rrrr} -1 & 4 & 15 & 18 & 7 \\ & & -4 & -11 & -7 \\ \hline & \boxed{4} & \boxed{11} & \boxed{7} & 0 \end{array}
$$

$\therefore$ ...... Proof by induction.

(ix) Prove that $\forall\, n \in \mathbb{N}$, $\displaystyle\sum_{r=1}^{n} \frac{3r+2}{r(r+1)(r+2)}(-2)^{r-1} = \frac{1}{2} - \frac{(-2)^n}{(n+1)(n+2)}$

$\underline{n=1}$ LHS $\displaystyle\sum_{r=1}^{1} \frac{3r+2}{r(r+1)(r+2)} = \frac{3\times1+2}{1(1+1)(1+2)} = \frac{5}{6}$

RHS $\displaystyle\frac{1}{2} - \frac{(-2)}{(1+1)(1+2)} = \frac{1}{2} + \frac{2}{6} = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

Assume true for $n=k$

ie. assume $\displaystyle\sum_{r=1}^{k} \frac{3r+2}{r(r+1)(r+2)}(-2)^{r-1} = \frac{1}{2} - \frac{(-2)^k}{(k+1)(k+2)}$ ------ $(*)$

Required to prove $\displaystyle\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)}(-2)^{r-1} = \frac{1}{2} - \frac{(-2)^{k+1}}{(k+2)(k+3)}$

$\underline{n=k+1}$

LHS $\displaystyle\sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)}(-2)^{r-1} = \sum_{r=1}^{k} \frac{3r+2}{r(r+1)(r+2)}(-2)^{r-1} + \frac{3(k+1)+2}{(k+1)(k+2)(k+3)}(-2)^k$

$$= \frac{1}{2} - \frac{(-2)^k}{(k+1)(k+2)} + \frac{(3k+5)}{(k+1)(k+2)(k+3)}(-2)^k$$

$$= \frac{1}{2} - \frac{(-2)^k}{(k+1)(k+2)}\left[1 - \frac{(3k+5)}{(k+3)}\right]$$

$$= \frac{1}{2} - \frac{(-2)^k}{(k+1)(k+2)}\left[\frac{k+3-3k-5}{k+3}\right]$$

$$= \frac{1}{2} - \frac{(-2)^k}{(k+1)(k+2)}\left[\frac{-2k-2}{k+3}\right]$$

$$= \frac{1}{2} - \frac{(-2)^{k+1}}{\cancel{(k+1)}(k+2)}\frac{\cancel{(k+1)}}{(k+3)}$$

$$= \frac{1}{2} - \frac{(-2)^{k+1}}{(k+2)(k+3)}$$

True for $n=1$.

Given true for $n=k \Rightarrow$ true for $n=k+1$.

$\therefore$ Proof by induction.

(x) Prove that if $f_r(x) = \dfrac{x(x+1)\ldots(x+r-1)}{r!}$ then, $\forall n \in \mathbb{N}$, $\displaystyle\sum_{r=1}^{n} f_r(x) = f_n(x+1) - 1$

$\underline{n=1}$ LHS $\displaystyle\sum_{r=1}^{1} f_r(x) = f_1(x) = \frac{x}{1!} = x$

RHS $f_1(x+1) - 1 = \dfrac{(x+1)}{1!} - 1 = x+1-1 = x$

$\therefore$ True for $n=1$.

Assume true for $n=k$.

i.e. assume $\displaystyle\sum_{r=1}^{k} f_r(x) = f_k(x+1) - 1$ ---------- $(*)$

Required to prove: $\displaystyle\sum_{r=1}^{k+1} f_r(x) = f_{k+1}(x+1) - 1 = \frac{(x+1)(x+2)\ldots(x+1+(k+1)-1)}{(k+1)!} - 1$

$= \dfrac{(x+1)(x+2)\ldots(x+k+1)}{(k+1)!} - 1$

$\underline{n=k+1}$ LHS $\displaystyle\sum_{r=1}^{k+1} f_r(x) = \sum_{r=1}^{k} f_r(x) + f_{k+1}(x)$

$$= f_k(x+1) - 1 + f_{k+1}(x)$$

$$= \frac{(x+1)(x+2)\ldots(x+1+k-1)}{k!} - 1 + \frac{x(x+1)\ldots(x+(k+1)-1)}{(k+1)!}$$

$$= \frac{(x+1)(x+2)\ldots(x+k)}{k!} + \frac{x(x+1)\ldots(x+k)}{(k+1)k!} - 1$$

$$= \frac{(x+1)(x+2)\ldots(x+k)}{k!}\left[1 + \frac{x}{k+1}\right] - 1$$

$$= \frac{(x+1)(x+2)\ldots(x+k)}{k!}\left[\frac{k+1+x}{k+1}\right] - 1$$

$$= \frac{(x+1)(x+2)\ldots(x+k)(x+k+1)}{(k+1)k!} - 1$$

$$= \frac{(x+1)(x+2)\ldots(x+k+1)}{(k+1)!} - 1$$

which is the RHS of what we were required to prove.

etc, etc, ... Proof by induction.

Q1(xi) Prove $\sum_{r=0}^{n} x^r(1+x)^{n-r} = (1+x)^{n+1} - x^{n+1} \quad \forall\, n \in \mathbb{N}$.

Ans 1(xi) <u>n=1</u> <u>LHS</u> $\sum_{r=0}^{1} x^r(1+x)^{1-r}$ <u>RHS</u> $(1+x)^{1+1} - x^{1+1}$

$$= x^0(1+x)^{1-0} + x^1(1+x)^{1-1} \qquad = (1+x)^2 - x^2$$

$$= 1(1+x) + x(1+x)^0 \qquad = 1+2x+x^2-x^2$$

$$= 1+x + x\times 1 \qquad = 1+2x$$

$$= 1+x+x$$

$$= 1+2x$$

LHS = RHS, so true for n=1

Assume true for n=k i.e. assume $\sum_{r=0}^{k} x^r(1+x)^{k-r} = (1+x)^{k+1} - x^{k+1}$ .....(1)

n=k+1: Required to prove $\sum_{r=0}^{k+1} x^r(1+x)^{k+1-r} = (1+x)^{k+1+1} - x^{k+1+1}$

$$= (1+x)^{k+2} - x^{k+2} \quad \text{-----} (2)$$

<u>n=k+1</u> <u>LHS</u> $\sum_{r=0}^{k+1} x^r(1+x)^{k+1-r} = \sum_{r=0}^{k} x^r(1+x)^{k+1-r} + (k+2)^{th}$ term

$$= \sum_{r=0}^{k} x^r(1+x)^{k-r}(1+x) + x^{k+1}(1+x)^{k+1-(k+1)}$$

$$= \left[(1+x)^{k+1} - x^{k+1}\right](1+x) + x^{k+1}(1+x)^0$$

$$= (1+x)^{k+2} - x^{k+1}(1+x) + x^{k+1}\times 1$$

$$= (1+x)^{k+2} - x^{k+1} - x^{k+2} + x^{k+1}$$

$$= (1+x)^{k+2} - x^{k+2}$$ which is (2), as required.

<u>Conclusion</u> It's true for n=1 ⇒ true for n=2 ⇒ true for n=3, etc, since assuming true for n=k ⇒ true for n=k+1

⇒ original statement true for all $n \in \mathbb{N}$.