Intermediate Mathematics

Proof by Induction

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The aim of this package is to provide a short self assessment programme for students who want to understand the method of proof by induction.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction (Summation)

Proof by induction involves statements which depend on the natural numbers, $n = 1, 2, 3, \ldots$. It often uses **summation notation** which we now briefly review before discussing induction itself.

We write the sum of the natural numbers up to a value n as:

$$1 + 2 + 3 + \cdots + (n-1) + n = \sum_{i=1}^{n} i\,.$$

The symbol $\sum$ denotes a sum over its argument for each natural number i from the lowest value, here $i = 1$, to the maximum value, here $i = n$.

Example 1: Write out explicitly the following sums:

a) $\displaystyle\sum_{i=3}^{6} i\,,$ b) $\displaystyle\sum_{i=1}^{3} (2i+1)\,,$ c) $\displaystyle\sum_{i=1}^{4} 2^i\,.$

The above sums when written out are:

a) $\sum_{i=3}^{6} i = 3 + 4 + 5 + 6\,,$

b) $\sum_{i=1}^{3} (2i + 1) = (2 \times 1 + 1) + (2 \times 2 + 1) + (2 \times 3 + 1) = 3 + 5 + 7\,,$

c) $\sum_{i=1}^{4} 2^i = 2^1 + 2^2 + 2^3 + 2^4\,.$

It is important to realise that the choice of symbol for the variable we are summing over is arbitrary, e.g., the following two sums are identical:

$$\sum_{i=1}^{4} i^3 = \sum_{j=1}^{4} j^3 = 1^3 + 2^3 + 3^3 + 4^3\,.$$

The variable that is summed over is called a **dummy variable**.

Quiz Select from the answers below the value of $\sum_{i=2}^{5} 2^i$.

(a) 1024, (b) 62, (c) 60, (d) 32.

EXERCISE 1. Expand the sums (click on the **green** letters for solutions):

(a) $\sum_{i=1}^{3}(2i-1)$, (b) $\sum_{j=1}^{4}(2j-1)$,

(c) $\sum_{s=1}^{4}10^s$, (d) $\sum_{j=1}^{3}12j$,

(e) $\sum_{i=0}^{3}3(2i+1)$, (f) $\sum_{j=1}^{3}\frac{1}{j^2}$.

EXERCISE 2. Express the following in summation notation.

(a) $1+20+400+8,000$, (b) $-3-1+1+3+5+7$.

Hint: write a) as a sum of powers of 20.

2. The Principle of Induction

Induction is an extremely powerful method of proving results in many areas of mathematics. It is based upon the following principle.

The Induction Principle: let $P(n)$ be a statement which involves a natural number n, i.e., $n = 1, 2, 3\dots$, then $P(n)$ is true for all n if

a) $P(1)$ is true, and

b) $P(k) \Rightarrow P(k+1)$ for all natural numbers k.

The standard *analogy* to this involves a row of dominoes: if it is shown that toppling one domino will make the next fall over (step b) and that the first domino has fallen (step a) then it follows that all of the dominoes in the row will fall over.

Example 2: the result of adding the first n natural numbers is:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\,; \quad \text{i.e., } P(n) \text{ is } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\,.$$

This is proven on the next page.

Step **a)** (the check): for $n = 1$, $\sum_{i=1}^{1} i = 1 = \dfrac{1 \times 2}{2}$. $\checkmark$

Step **b)** (the induction step): assume the result is true for $n = k$, i.e., assume

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\,.$$

The sum for $n = k + 1$ may be written

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)\,.$$

Using the assumption this becomes

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \frac{k(k+1)}{2} + (k+1)\\ &= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}\,. \quad \checkmark\end{aligned}$$

which is the desired result for $n = k + 1$.

It is always worth making the check, step **a)**, first. If it does not work one knows that the result must be wrong and saves time.

Example 3: the result of adding the odd natural numbers is:

$$\begin{aligned} 1 &= 1\,, \\ 1+3 &= 4\,, \\ 1+3+5 &= 9\,, \\ 1+3+5+7 &= 16\,, \\ 1+3+5+7+9 &= 25\,. \end{aligned}$$

This seems to indicate that

$$\sum_{j=1}^{n}(2j-1) = n^2\,.$$

We will now use induction to prove this result.

Step **a)** (the check): we have already seen the initial step of the proof, i.e., for $n = 1$, $\sum_{j=1}^{1}(2j-1) = 1 = 1^2$. ✓

Step **b)** (the inductive step): we *assume* it is true for $n = k$, i.e., assume

$$\sum_{j=1}^{k}(2j-1) = k^2 \, .$$

and need to *show* that it follows that $\sum_{j=1}^{k+1}(2j-1) = (k+1)^2$.

Write this sum (over $k+1$ terms) as a sum over the first k terms *plus* the final term (where $j = k+1$)

$$\begin{aligned}
\sum_{j=1}^{k+1}(2j-1) &= \sum_{j=1}^{k}(2j-1) + (2\times(k+1)-1) \\
&= k^2 + (2(k+1)-1) \quad \textit{from the assumption} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2 \, . \quad \checkmark
\end{aligned}$$

This completes step **b)** and by the **Principle of Induction** we have proven the result.

EXERCISE 3. Use the **Principle of Induction** to prove the following results. Unless stated otherwise assume n is a natural number. (Click on the **green** letters for the solutions.)

(a) $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

(b) $$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4},$$

(c) $$\sum_{j=1}^{n} 2^{j-1} = 2^n - 1,$$

(d) $$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}, \qquad \text{for } x \neq 1 \text{ and integers } n \geq 0.$$

Hint: in the last exercise the check must be performed for $n = 0$.

Induction can also be used to prove a great many other results. The next section treats some further applications.

3. Further Examples

Example 3: for n a natural number prove that:

1) if $n \geq 2$, then $n^3 - n$ is always divisible by 3, **2)** $n < 2^n$.

1) If a number is divisible by 3 it can be written as $3r$ for integer r

Step **a)** (check): for $n = 2$, $2^3 - 2 = 6 = 3 \times 2$; so divisible by 3. ✓

Step **b)** (induction step): assume that it is true for $n = k$, i.e., *assume* that $k^3 - k = 3r$. For $n = k + 1$ we have:

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 3r + 3k^2 + 3k \qquad \textit{using the assumption} \\ &= 3(r + k^2 + k)\,. \quad \checkmark \end{aligned}$$

The Principle of Induction thus implies that $n^3 - n$ is indeed divisible by 3 for all $n \geq 2$.

2) Show by induction that $n < 2^n$ for all natural numbers n.

Step **a)** (check): for $n = 1$, since $2^1 = 2$, it is true that $1 < 2^1$. ✓

Step **b)** (induction step): assume it is true for $n = k$, i.e., $k < 2^k$. Then

$$k + 1 < 2^k + 1\,,$$

since $1 < 2^k$ for any natural number k and this implies that

$$\begin{array}{rrcl} & k+1 & < & 2^k + 2^k = 2 \times 2^k = 2^{k+1} \\ \text{Hence} & k+1 & < & 2^{k+1}\,. \quad \checkmark \end{array}$$

From the Principle of Induction, $n < 2^n$ for any natural number n.

Quiz Which of the following properties is *not* necessary for a natural number n to be divisible by 10?

(a) 10 divides n^2,
(b) 20 divides $4n$,
(c) 5 divides $n/2$,
(d) 100 divides $2n$.

Exercise 4. Use the **Principle of Induction** to prove the following results. Assume n is a natural number. (Click on the **green** letters for the solutions.)

(a) $5^n - 1$ is divisible by 4,

(b) $9^n + 3$ is divisible by 4,

(c) $3^n > n^2$,

(d) $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$.

Exercise 5. Consider the following sequence of sums:

$$\frac{1}{1\times 2}\,; \qquad \frac{1}{1\times 2}+\frac{1}{2\times 3}\,; \qquad \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}\,; \qquad \cdots$$

(a) Calculate the sums and guess the general pattern.
(b) Prove your conjecture using proof by induction.

4. Final Quiz

Begin Quiz

1. Select the sum which is *not* equivalent to $\sum_{i=1}^{15} 25i$.

(a) $25\sum_{i=1}^{15} i$, (b) $\sum_{j=1}^{15} 25j$, (c) $\sum_{i=1}^{75} 5i$, (d) $5\sum_{j=1}^{15} 5j$.

2. Select the expression below which is meaningless:

(a) $\sum_{j=1}^{n} j^j$, (b) $\sum_{i=1}^{i} 2i$, (c) $n\sum_{i=1}^{n} i^n$, (d) $\sum_{i=n}^{2n} i$.

3. In a proof by induction that $6^n - 1$ is divisible by 5, which result may occur in the inductive step (let $6^k - 1 = 5r$)? (N.B. you may need to re-arrange your equation.)

(a) $6^{k+1} - 1 = 5(6r - 1)$, (b) $6^{k+1} - 1 = 6(5r + 1)$,

(c) $6^{k+1} - 1 = 5(6r + 1)$, (d) $6^{k+1} - 1 = 5 \times 6r + 7$.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The sum $\sum_{i=1}^{3}(2i-1)$ may be written as

$$\begin{aligned}\sum_{i=1}^{3}(2i-1) &= (2\times 1-1)+(2\times 2-1)+(2\times 3-1)\\ &= 1+3+5\\ &= 9\,.\end{aligned}$$

One sees that the sequence $2i-1$, where i is an integer, generates the odd integers.

Click on the **green** square to return

$\square$

Exercise 1(b) We can write:

$$\sum_{j=1}^{4}(2j-1) = \sum_{j=1}^{3}(2j-1) + (2\times 4-1)\,,$$

and from the previous exercise we have shown that

$$\sum_{i=1}^{3}(2i-1) = 9\,.$$

Since i is a dummy variable it could just as well be called j, thus

$$\sum_{j=1}^{4}(2j-1) = 9 + (2\times 4-1) = 16\,.$$

Note that this result, and the last one, are perfect squares! We will prove a general version of this below.

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Exercise 1(c) The series $\sum_{s=1}^{4} 10^s$ is

$$\begin{aligned} \sum_{s=1}^{4} 10^s &= 10^1 + 10^2 + 10^3 + 10^4 \\ &= 11,110\,. \end{aligned}$$

Click on the **green** square to return $\square$

Exercise 1(d) The series $\sum_{j=1}^{3} 12j$ may be written as

$$\begin{aligned}\sum_{j=1}^{3} 12j &= (12 \times 1) + (12 \times 2) + (12 \times 3) \\ &= 12 \times (1 + 2 + 3)\,,\end{aligned}$$

which shows us that we could have simply extracted the constant factor

$$\sum_{j=1}^{3} 12j = 12 \sum_{j=1}^{3} j$$

The numerical value of the series is

$$\sum_{j=1}^{3} 12j = 12 \times (1 + 2 + 3) = 72\,.$$

Click on the **green** square to return $\square$

Exercise 1(e) To calculate the series $\sum_{i=0}^{3} 3(2i+1)$ it is simplest to extract the common factor 3:

$$\begin{aligned}
\sum_{i=0}^{3} 3(2i+1) &= 3\sum_{i=0}^{3}(2i+1) \\
&= 3\times\{(2\times 0+1)+(2\times 1+1) \\
&\qquad +(2\times 2+1)+(2\times 3+1)\} \\
&= 3\{1+3+5+7\} \\
&= 48\,.
\end{aligned}$$

Note that the sequence $2i+1$ also generates the odd integers.

Click on the **green** square to return □

Exercise 1(f) The series $\sum_{j=1}^{3} \frac{1}{j^2}$ may be written as

$$\begin{aligned}
\sum_{j=1}^{3} \frac{1}{j^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \\
&= 1 + \frac{1}{4} + \frac{1}{9} \\
&= 1 + \frac{9+4}{36} \\
&= 1 + \frac{13}{36} \\
&= \frac{36+13}{36} \\
&= \frac{49}{36}\,.
\end{aligned}$$

Click on the **green** square to return □

Exercise 2(a) To write $1 + 20 + 400 + 8,000$ as a sum, note that

$$\begin{aligned} 1 &= 20^0\,, \\ 20 &= 20^1\,, \\ 400 &= 20^2\,, \\ 8,000 &= 20^3\,. \end{aligned}$$

This shows that we may write

$$1 + 20 + 400 + 8,000 = \sum_{j=0}^{3} 20^j\,.$$

or, alternatively,

$$1 + 20 + 400 + 8,000 = \sum_{i=1}^{4} 20^{i-1}\,.$$

Click on the **green** square to return □

Exercise 2(b) The sum $-3-1+1+3+5+7$ is a sum over a consecutive range of odd integers. It may be written in many different ways. Here are three possibilities.

$$-3-1+1+3+5+7=\sum_{j=-1}^{4}(2j-1)\,,$$

or

$$\sum_{i=1}^{6}[(2j-1)-4]\,.$$

or

$$\sum_{j=-4}^{3}(2j+1)\,.$$

There are many other possibilities.

Click on the **green** square to return

$\square$

Exercise 3(a) We are asked to prove that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Step **a)**, **check** when $i = 1$: $\displaystyle\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{1(1+1)(2\times 1+1)}{6}$. ✓

Step **b)** **assume** the result for $n = k$, so for $n = k+1$:

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \textit{assumption} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \textit{extract common factor } (k+1) \\
&= \frac{(k+1)[2k^2+k+6k+6]}{6} = \frac{(k+1)[2k^2+7k+6]}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)(2[k+1]+1)}{6} . && ✓
\end{aligned}
$$

Click on the **green** square to return □

Exercise 3(b) To prove that $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$,

first **check** it for $n = 1$: i.e., $\sum_{j=1}^{1} j^3 = 1 = \dfrac{1^2 \times 2^2}{4}$. ✓

Inductive step: assume it is true for $n = k$, then for $n = k + 1$

$$
\begin{aligned}
\sum_{j=1}^{k+1} j^3 &= \sum_{j=1}^{k} j^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} = \frac{(k+1)^2(k+2)^2}{4}. \quad \checkmark
\end{aligned}
$$

which, by the Principle of Induction, proves the result.

Click on the **green** square to return □

Exercise 3(c) To prove that $\displaystyle\sum_{j=1}^{n} 2^{j-1} = 2^n - 1\,,$

first **check** it for $n = 1$: i.e., $\sum_{j=1}^{1} 2^{1-1} = 1 = 2^1 - 1\,.$ ✓

In the **inductive step** first assume it is true for $n = k$, then for $n = k + 1$

$$\begin{aligned}\sum_{j=1}^{k+1} 2^{j-1} &= \sum_{j=1}^{k} 2^{j-1} + 2^{k+1-1} \\ &= 2^k - 1 + 2^k \\ &= 2 \times 2^k - 1 = 2^{k+1} - 1\,. \quad \checkmark\end{aligned}$$

which, by the Principle of Induction, proves the result.

Click on the **green** square to return □

Exercise 3(d) To prove that $\displaystyle\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}\,, \quad$ for $x \neq 1$

first **check** it for $n = 0$: i.e., $\sum_{j=0}^{0} x^j = x^0 = 1 = \dfrac{1-x^{0+1}}{1-x}\,. \quad \checkmark$

In the **inductive step** assume it is true for $n = k$, so for $n = k+1$

$$\begin{aligned}
\sum_{j=0}^{k+1} x^j &= \sum_{j=0}^{k} x^j + x^{k+1} \\
&= \frac{1-x^{k+1}}{1-x} + x^{k+1} \\
&= \frac{1-x^{k+1} + (1-x)(x^{k+1})}{1-x} \\
&= \frac{1-x^{k+2}}{1-x}\,. \quad \checkmark
\end{aligned}$$

which, by the Principle of Induction, proves the result.

Click on the **green** square to return $\square$

Exercise 4(a) To prove that $5^n - 1$ is divisible by 4, first **check** it for $n = 1$: i.e., $5^1 - 1 = 5 - 1 = 4 = 4 \times 1$. ✓

Inductive step: assume it is true for $n = k$, i.e., *assume* $5^k - 1 = 4r$, where r is an integer. Then for $n = k + 1$

$$\begin{aligned} 5^{k+1} - 1 &= 5 \times 5^k - 1 \\ &= 5 \times (4r + 1) - 1\,, \quad \textit{since } 5^k = 4r + 1 \\ &= 4 \times 5r + 5 - 1 \\ &= 4 \times (5r + 1)\,. \quad \checkmark \end{aligned}$$

which, from the Principle of Induction, proves the result.

Click on the **green** square to return

□

Exercise 4(b) To prove that $9^n + 3$ is divisible by 4, first **check** it for $n = 1$: i.e., $9^1 + 3 = 9 + 3 = 12 = 4 \times 3$. ✓

Inductive step: assume it is true for $n = k$, i.e., *assume* $9^k + 3 = 4r$, where r is an integer. Then for $n = k + 1$

$$\begin{aligned} 9^{k+1} + 3 &= 9 \times 9^k + 3 \\ &= 9 \times (4r - 3) + 3 \\ &= 4 \times 9r - 27 + 3 \\ &= 4 \times 5r - 24 \\ &= 4 \times (5r - 6)\,. \quad \checkmark \end{aligned}$$

which, from the Principle of Induction, proves the result.

Click on the **green** square to return

□

Exercise 4(c) To prove that $3^n > n^2$, i.e., $3^n - n^2 > 0$, using proof by induction, we first **check** it for $n = 1$: i.e., $3^1 = 3 > 1^2 = 1\,.$ ✓

Inductive step: now assume it is true for $n = k$, i.e., *assume* that $3^k - k^2 > 0$. For $n = k + 1$

$$\begin{aligned}
3^{k+1} - (k+1)^2 &= 3 \times 3^k - (k+1)^2 \\
&= 3(3^k - k^2 + k^2) - (k+1)^2 \\
&= 3(3^k - k^2) + 3k^2 - k^2 - 2k - 1 \\
&= 3(3^k - k^2) + k^2 + k^2 - 2k - 1 \\
&= 3(3^k - k^2) + k^2 + (k-1)^2 - 2\,.
\end{aligned}$$

The first term is positive by assumption and the second and third terms are both squares so they cannot be negative. If $k \geq 2$ then $k^2 - 2$ is positive. Thus $3^{k+1} - (k+1)^2 > 0$. From the Principle of Induction this proves the result.

Click on the **green** square to return □

Exercise 4(d) We have to prove that $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$.

Check: for $n = 1$ one has: $\displaystyle\sum_{j=1}^{1} \frac{1}{j^2} = 1 \leq 2 - \frac{1}{1} = 1$. ✓

Inductive step: assume it is true for $n = k$, then for $n = k + 1$

$$
\begin{aligned}
\sum_{j=0}^{k+1} \frac{1}{j^2} &= \sum_{j=0}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \textit{assumption!} \\
&\leq 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \leq 2 - \frac{k^2 + k + 1}{k(k+1)^2} && \textit{inequality preserved by dropping } 1 \textit{ in numerator} \\
&\leq 2 - \frac{k(k+1)}{k(k+1)^2} \leq 2 - \frac{1}{k+1}. \quad ✓
\end{aligned}
$$

which, by the Principle of Induction, proves the result.

Click on the **green** square to return □

Exercise 5(a) The sums are

$$\begin{aligned} \frac{1}{1\times 2} &= \frac{1}{2}\,, \\ \frac{1}{1\times 2}+\frac{1}{2\times 3} = \frac{1}{2}+\frac{1}{6} &= \frac{2}{3}\,, \\ \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4} = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} &= \frac{3}{4}\,. \end{aligned}$$

It is thus *natural to guess* that in general:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)n} = \frac{n-1}{n}\,.$$

Click on the **green** square to return

Exercise 5(b) We have **already checked** that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)n}=\frac{n-1}{n}\,.$$

for $n=2\,,\ n=3$ and $n=4$. Now assume it is true for $n=k$. Then, in the **inductive step**, consider $n=k+1$ where we have:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(k-1)k}+\frac{1}{k(k+1)}\,.$$

Using the assumption this may be written as:

$$\frac{k-1}{k}+\frac{1}{k(k+1)}\,,$$

which is

$$\frac{k-1}{k}+\frac{1}{k(k+1)}=\frac{(k-1)(k+1)+1}{k(k+1)}=\frac{k^2}{k(k+1)}=\frac{k}{k+1}\,.\ \checkmark$$

From the **Principle of Induction**, the result is proven.

Click on the **green** square to return

□

Solutions to Quizzes

Solution to Quiz: The sum

$$\sum_{i=2}^{5} 2^i = 2^2 + 2^3 + 2^4 + 2^5 \,,$$

since the lowest value of i is 2. This implies

$$\sum_{i=2}^{5} 2^i = 4 + 8 + 16 + 32 = 60 \,.$$

End Quiz

Solution to Quiz: If n is divisible by 10, then $\dfrac{n}{10} = r$ is an integer.

First let us show that **(a)**, **(b)** and **(c)** are all necessary.

(a) Since $n = 10r$, $n^2 = 10r \times 10r = 100r^2$ which is divisible by 10.

(b) Since $n = 10r$, $4n = 4 \times 10r = 40r$ which is divisible by 20.

(b) Since $n = 10r$, $\dfrac{n}{2} = 5r$ which is divisible by 5.

Finally to demonstrate that **(d)** is **not** necessary, we only need to find an example for which it is false. Let $n = 10$, then 10 divides n, but $2n = 20$ is not divisible by 100.

This is an example of *proof by contradiction.*

End Quiz