SCHOLAR Study Guide

SQA Advanced Higher Mathematics Unit 3

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1. Advanced Higher Mathematics

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Topic 1

Vectors

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Learning Objectives

- *Use vectors in three dimensions.*

Minimum Performance Criteria:

- *Calculate a vector product.*
- *Find the equation of a line in parametric form.*
- *Find the equation of a plane in Cartesian form given a normal and a point in the plane.*

Prerequisites

A sound knowledge of the following subjects is required for this topic:

- Plotting points in three dimensions.
- Simple trigonometry.
- Algebraic manipulation.
- Equations of straight lines.
- Gaussian elimination

1.1 Revision exercise

Learning Objective

Identify areas that need revision

Revision exercise

15 min

If any question in this revision exercise causes difficulty then it may be necessary to revise the techniques before starting the topic. Ask a tutor for advice.

Q1: Draw and label a three-dimensional graph with axes x, y and z

Q2: Find the equation of the line which joins the two points (4, 5) and (-1, -10)

Q3: Expand fully the following: (ab + 2c)(ac - b)(c + b)

Hence simplify the expression if $a^2 = 2$, $b^2 = 1$ and $c^2 = 1$

Q4: Use Gaussian elimination to solve the equations

$7x - 7y - 2z = 8$

$2x - y - 3z = -6$

$2x + 3y - z = -4$

1.2 Introduction

Learning Objective

Define basic vector terms

Quantities that have magnitude only are called **scalars**.

Weights, areas and volumes are all examples of scalars.

Many quantities are not sufficiently defined by their magnitudes alone. For example, a movement or displacement from a point P to a point Q requires both the distance between the points and the direction from P to Q.

Example If Q is 45 km North West of P then the distance is 45 km and the direction is North West.

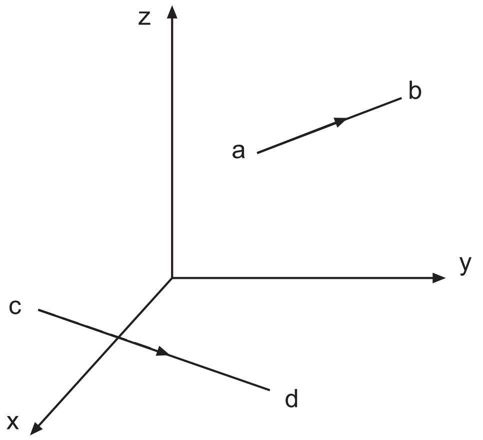
There is a special type of quantity to cope with this. It is called a vector quantity.

Vector quantity

A vector quantity is a quantity which has both direction and magnitude.

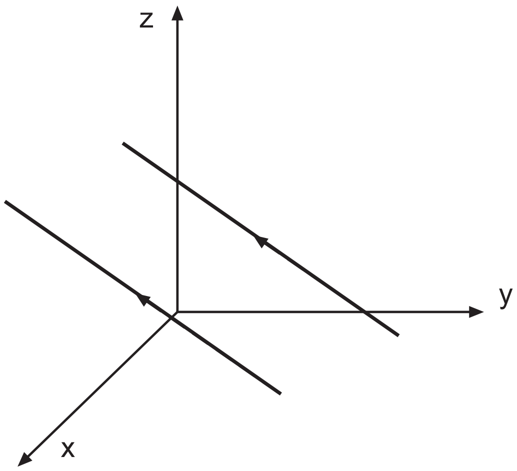
Geometrically, a vector can be represented by a directed line segment.

The two lines ab and cd represent two vectors in three dimensions.



Parallel vectors

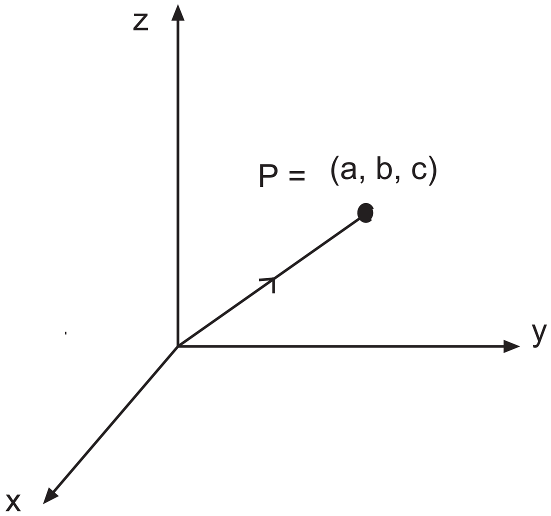
Parallel vectors have the same direction but the magnitudes are scalar multiples of each other.



There is a particular type of vector called a position vector.

Position vector

A position vector is a vector which starts at the origin.



A vector from the origin O, for example, to the point P, may be expressed by a small letter in the form **p** or $\underset{\sim}{p}$ or by the directed vector $\overrightarrow{OP}$. It is customary for textbooks to use the bold form and this will be the style in this topic.

A position vector can also be expressed as an ordered triple.

For example, the vector in the previous diagram from the origin to the point P with coordinates (a, b, c) can be expressed as the ordered triple $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Note that the vector from the origin to the point P has a magnitude (length) and direction from the origin to the point (a, b, c)

Length of a vector

Let **p** be the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then the length of **p** is defined as $|\mathbf{p}| = \sqrt{a^2 + b^2 + c^2}$

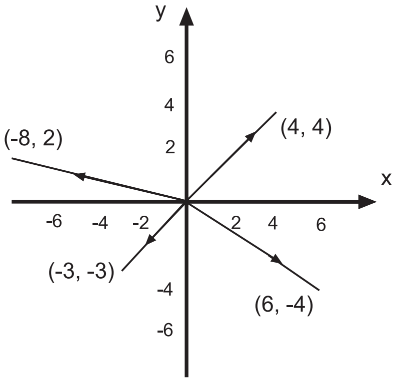
Examples

1. Find the length of **a** = $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

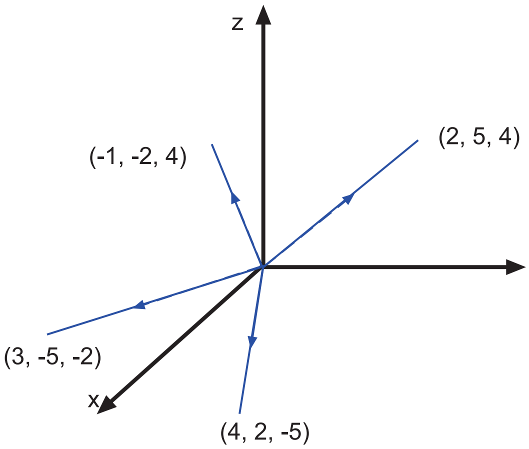
Answer:

The length is given by $|\mathbf{a}| = \sqrt{2^2 + 3^2 + 4^2} = 5.38516$

2. These are all position vectors in 2 dimensions.



These are position vectors in 3 dimensions.

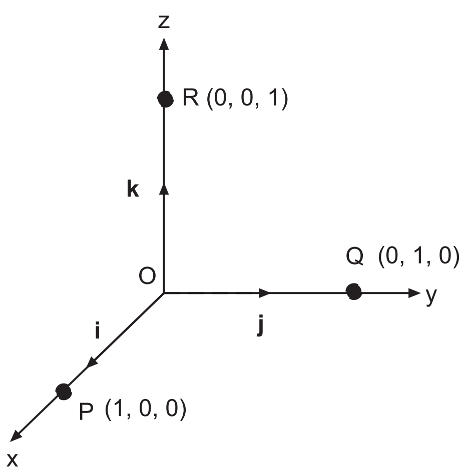


In this topic most of the work will be covered using position vectors. The word 'position' will be omitted unless confusion is likely.

Unit vector

As the name would suggest a unit vector is a vector of magnitude 1. That is, the length (or magnitude) of the vector is 1

There are 3 special unit vectors which lie along the x, y and z axes. The point P has coordinates (1, 0, 0), Q has (0, 1, 0) and R has (0, 0, 1)



Each of the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ is one unit long and is denoted by **i**, **j**, and **k** respectively. **i**, **j** and **k** are called the **standard basis vectors**.

Thus **i** is the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

similarly **j** is the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

and **k** is the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Every vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ can be written in component form as p**i** + q**j** + r**k** Conversely every vector in component form p**i** + q**j** + r**k** can be written as $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

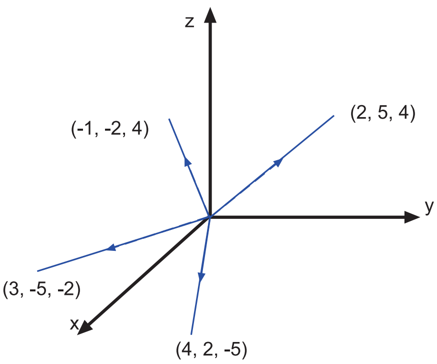
Example The vector **p** = $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$ = 3**i** - 5**j** + 2**k**

Vector length exercise

10 min

There is another version of this exercise on the web for you to try if you prefer it.

Q5: Find the lengths of the position vectors in the diagram.



Q6: Find the length of the vector $\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

Vector notation exercise

5 min

There is another version of this exercise on the web for you to try if you like.

Q7: Express the vector $\begin{pmatrix} -3 \\ 3 \\ -4 \end{pmatrix}$ using the standard basis vectors.

Q8: Express the vector **i** + 2**j** + **k** as an ordered triple.

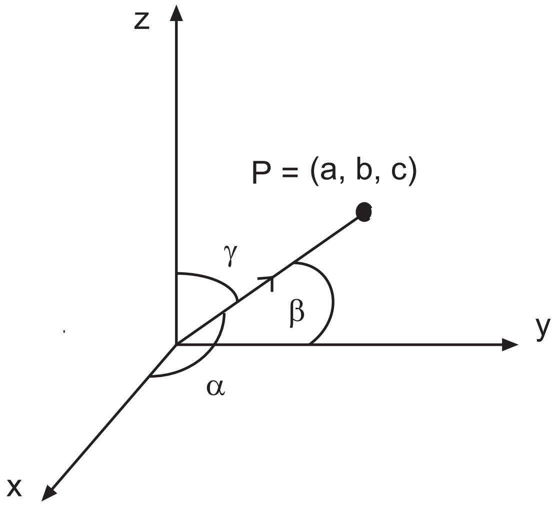
1.3 Direction ratios and cosines

Learning Objective

Find direction ratios and direction cosines of a vector

Let **p** be the vector a**i** + b**j** + c**k**

This can be represented on a diagram as follows.



Notice that the vector **p** makes an angle of α with the x-axis, an angle of β with the y-axis and an angle of γ with the z-axis.

Thus

cos α = $\frac{a}{|\mathbf{p}|}$

cos β = $\frac{b}{|\mathbf{p}|}$

cos γ = $\frac{c}{|\mathbf{p}|}$

These values $\frac{a}{|\mathbf{p}|}$, $\frac{b}{|\mathbf{p}|}$ and $\frac{c}{|\mathbf{p}|}$ are called the **direction cosines** of the vector **p**

The ratios of a : b : c are called the **direction ratios** of the vector **p**

Examples

1. Find the direction ratios of the vector -2**i** - 3**j** + 4**k**

Answer:

The direction ratios are -2 : -3 : 4

2. Find the direction cosines of the vector **p** = 2**i** - 2**j** - **k**

Answer:

| **p** | = 3

The direction cosines are $^2/_3$, $^{-2}/_3$ and $^{-1}/_3$

Direction ratios and direction cosines exercise

5 min

There is a web exercise similar to this for you to try if you prefer it.

Q9: Find the direction ratios of the following vectors:

a) -3**i** + 2**j** - **k**
b) **i** + **j** + **k**
c) 2**i** - **j** - **k**

Q10: Find the direction cosines of the following vectors:

a) 2**i** + 3**j** + 6**k**
b) **i** - **j** - **k**
c) 4**i** - 4**j** + 2**k**

1.4 Basic arithmetic and scalar product of vectors

Learning Objective

Add, subtract, multiply and find the scalar product of vectors in three dimensions

These techniques are straightforward and the following examples will demonstrate them.

Vectors in 2d addition and subtraction

10 min

There is a web animation to review addition and subtraction in 2 dimensions.

1.4.1 Arithmetic on vectors in three dimensions

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are vectors then **p + q** is the vector $\begin{pmatrix} a + x \\ b + y \\ c + z \end{pmatrix}$

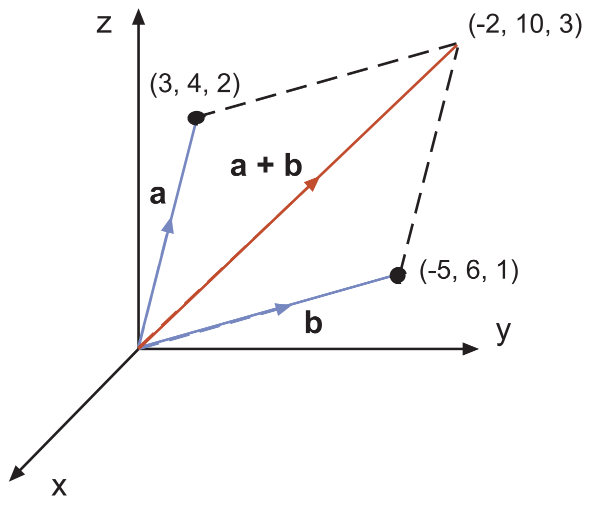
The diagram in the following example shows how this is related to the parallelogram.

Example Add the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix}$

Answer:

The sum is found by adding the corresponding values so

$$\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 + (-5) \\ 4 + 6 \\ 2 + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 3 \end{pmatrix}$$



If **p** and **q** are written in terms of the standard bases then

p = a**i** + b**j** + c**k**, **q** = x**i** + y**j** + z**k** and **p + q** = (a + x)**i** + (b + y)**j** + (c + z)**k**

Example Add the vectors -2**i** + 3**j** - 4**k** and -3**i** - 6**j** - 3**k**

Answer:

Adding the corresponding values gives

(-2**i** + 3**j** - 4**k**) + (-3**i** - 6**j** - 3**k**)

= (-2 - 3)**i** + (3 - 6)**j** + (-4 - 3)**k**

= -5**i** - 3**j** - 7**k**

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are vectors then **p - q** is the vector $\begin{pmatrix} a - x \\ b - y \\ c - z \end{pmatrix}$

Examples

1. Evaluate $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

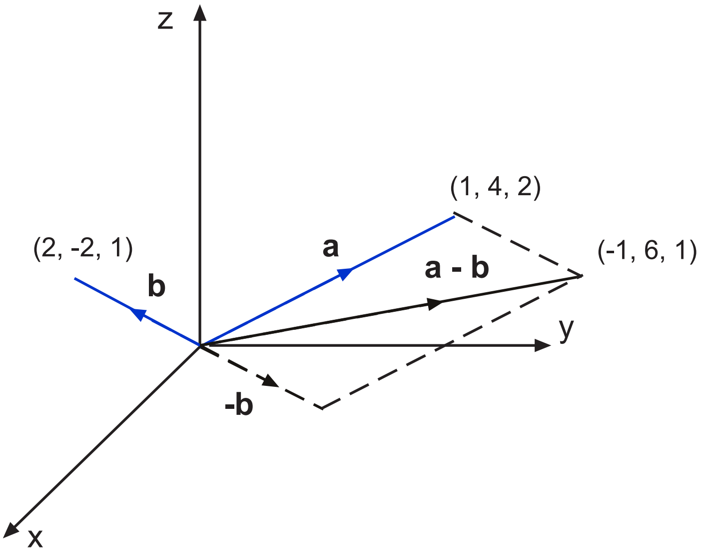
Answer:

$$\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2 \\ 4-(-2) \\ 2-1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$$

The following diagram shows how this relates to a parallelogram. Note that in subtraction, the vector **-b** instead of **b** forms one of the sides (**a** + (**-b**) = **a - b**)



It is also possible to subtract vectors that are written in terms of the standard bases.

2. Evaluate (**i** + 4**j** + 2**k**) - (2**i** -2**j** + **k**)

Answer:

(**i** + 4**j** + 2**k**) - (2**i** - 2**j** + **k**)

= **i** + 4**j** + 2**k** - 2**i** + 2**j** - **k**

= -**i** + 6**j** + **k**

Vectors can also be multiplied by a scalar.

If **p** = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and λ is a real number, (a scalar) then λ**p** = $\begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$

Example If **a** = $\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$ find -2**a**

Answer:

$$-2\mathbf{a} = -2 \times \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \times -2 \\ -2 \times 3 \\ -2 \times -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$$

Using the standard bases, the multiplication is just as straightforward.

Example If **a** = 3**i** - **j** + 2**k** find -4**a**

Answer:

Each of the terms is multiplied by -4 to give

-4**a** = -12**i** + 4**j** - 8**k**

This is particularly useful for determining whether two vectors are parallel or not. Recall that two vectors which are parallel have the same direction but their magnitudes are scalar multiples of each other.

Example Show that the two vectors **a** = $\begin{pmatrix} -3 \\ 5 \\ 3 \end{pmatrix}$ and **b** = $\begin{pmatrix} 6 \\ -10 \\ -6 \end{pmatrix}$ are parallel.

Answer:

Each component of **b** is -2 times the corresponding component of **b**

That is, **b** = -2**a** and the vectors are parallel.

Basic skills exercise

10 min

There is another exercise on the web for you to try if you prefer it.

Q11: Add the vectors 2**i** + 3**j** - 2**k** and **i** - 3**j** - **k**

Q12: Add $\begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$ and give the answer as an ordered triple.

Q13: Subtract 3**i** - 2**j** + 5**k** from -2**i**

Q14: Subtract $\begin{pmatrix} -7 \\ -2 \\ 4 \end{pmatrix}$ from $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ and give the answer as an ordered triple.

Q15: Show that 2**i** - **j** + 3**k** and -4**i** + 2**j** - 6**k** are parallel vectors.

Q16: Find a vector parallel to 2**i** - 4**j** + **k** with z component equal to 3

1.4.2 Scalar product of vectors in three dimensions

There are two ways of expressing the scalar product. The first is algebraically in component form.

Scalar product in component form

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

then the scalar product is the number $\mathbf{p} \bullet \mathbf{q} = ad + be + cf$

It is important to note that the scalar product of two vectors is not a vector. It is a scalar. The scalar product is also known as the dot product.

Example Find the scalar product of $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

Answer:

$\mathbf{a} \bullet \mathbf{b} = -6 + 2 + 0 = -4$

The scalar products of the standard basis vectors are useful to note:

$\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{i} \bullet \mathbf{k} = 0$

also

$\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{i}$

$\mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{j}$

$\mathbf{i} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i}$

These results can be used to give the scalar product in **i**, **j**, **k** notation of the two vectors $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

Example Find $(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \bullet (3\mathbf{i} + 2\mathbf{j})$ by multiplying out the brackets and using the properties of the vector product on standard bases.

Answer:

$(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \bullet (3\mathbf{i} + 2\mathbf{j})$

$= -6\mathbf{i} \bullet \mathbf{i} - 4\mathbf{i} \bullet \mathbf{j} + 3\,\mathbf{j} \bullet \mathbf{i} + 2\mathbf{j} \bullet \mathbf{j} + 9\mathbf{k} \bullet \mathbf{i} + 6\mathbf{k} \bullet \mathbf{j}$

$= -6 - \mathbf{i} \bullet \mathbf{j} + 2 + 9\mathbf{i} \bullet \mathbf{k} + 6\mathbf{j} \bullet \mathbf{k}$

$= -6 + 2 = -4$

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10 min

Algebraic scalar product exercise

There is an exercise similar to this on the web for you to try if you wish.

Q17: Find **a** • **b** where $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$

Q18: Find the scalar product of the vectors 2**i** + 3**j** - 2**k** and **i** - 3**j** - **k**

Q19: Find the scalar product of the vectors $\begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$

Algebraic rules of scalar products

There are several useful properties of scalar products.

Property 1
$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

Details of the proof can be found in the Proof section at Proof 3.

Property 2
$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

The proof of this result can be found in the section headed Proofs at Proof 4.

Property 3
$\mathbf{a} \bullet \mathbf{a} = \lvert \mathbf{a} \rvert^2 \geqslant 0$

The proof of this result can be found in the section headed Proofs at Proof 5.

Property 4
$\mathbf{a} \bullet \mathbf{a} = 0$ if and only if $\mathbf{a} = 0$

The proof of this result can be found in the section headed Proofs at Proof 6.

Q20: Show that the property **a** • (**b** + **c**) = **a** • **b** + **a** • **c** holds for the three vectors **a** = 2**i** - **k**, **b** = -3**i** - **j** + 3**k** and **c** = -**i** + 2**j** - **k**

Q21: Using $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ prove **a** • **b** = **b** • **a**

The second way of expressing the scalar product is geometrically.

Scalar product in geometric form

The scalar product of two vectors **a** and **b** is defined as

$\mathbf{a} \bullet \mathbf{b} = \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \cos \theta$ where θ is the angle between **a** and **b**, $0 \leq \theta \leq 180°$

The proof of this is shown at Proof 1 in the section headed Proofs.

Example Find the scalar product of the vectors **a** and **b** where the length of **a** is 5, the length of **b** is 4 and the angle between them is 60°

Answer:

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = 5 \times 4 \times \cos 60° = 10$

The geometric form of the scalar product is especially useful to find angles between vectors.

Example Find the angle between **i** + 4**j** and -8**i** + 2**j**

Answer:

$\mathbf{a} \bullet \mathbf{b} = -8 + 8 = 0$

$|\mathbf{a}| = \sqrt{17}$

$|\mathbf{b}| = \sqrt{68}$

$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = 0$ so $\theta = \frac{\pi}{2}$

The last example demonstrates an important geometric property of the scalar product.

Property 5
For non-zero vectors, **a** and **b** are perpendicular if and only if **a** • **b** = 0

In this case, **a** and **b** are said to be orthogonal vectors.

The proof of this result is shown in the section headed Proofs at Proof 7.

To show that two vectors are perpendicular however, the algebraic form is all that is required.

Example Show that the two vectors **a** = -2**i** + 2**j** + **k** and **b** = 3**i** + 2**j** + 2**k** are perpendicular.

Answer:

$\mathbf{a} \bullet \mathbf{b} = (-2 \times 3) + (2 \times 2) + (1 \times 2) = 0$

The vectors are perpendicular.

Geometric scalar product exercise

There is another exercise on the web for you to try if you prefer it.

10 min

Q22: Find the scalar product of two vectors that have lengths 4 and 8 units respectively and an angle of 45° between them. Give the answer to 3 decimal places.

Q23: Find the scalar product of two vectors whose lengths are 2.5 and 5 and have an angle of 150° between them. Give the answer to 3 decimal places.

Q24: Determine the angle between two vectors whose scalar product is -6 and whose lengths are 4 and 3

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Q25: Find the angle between the two vectors 3**i** - 4**j** and 12**i** + 5**j**. Give the angle in degrees correct to 2 decimal places.

Q26: Find the angle ACB if A is (0, 1, 6), B is (2, 3, 0) and C is (-1, 3, 4)

Perpendicular vectors exercise

10 min

There is an exercise on the web for you to try if you like.

Q27: Find c so that c**i** + 2**j** - **k** is perpendicular to **i** - 3**k**

Q28: Find c so that 3**i** + c**j** + 4**k** is perpendicular to 2**i** -2**j** + 3**k**

Q29: Show that the vectors -4**i** - 3**j** + 2**k** and 2**i** - 2**j** + **k** are perpendicular.

1.5 Vector product - algebraic form

Learning Objective

Calculate the vector product algebraically

Like the scalar product, there are two ways of expressing a vector product. The first is algebraically in component form.

Vector product in component form

If **p** = a**i** + b**j** + c**k** and **q** = d**i** + e**j** + f**k**

then the **vector product** is defined as the vector

p x q = (bf - ec)**i** - (af - dc)**j** + (ae - db)**k**

Hence, unlike the scalar product, the vector product of two vectors is another vector.

This is actually the **determinant** of the matrix $\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$

By putting the vectors in this form it is easier to see how the calculation is done and why the vector product is sometimes called the cross product.

Determinants of matrices will be studied in detail in topic 12 in this course.

Examples

1. If **a** = $\begin{pmatrix} 4 \\ -5 \\ -6 \end{pmatrix}$ and **b** = $\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$ find **a x b**

Answer:

To use the formula, the vectors **a** and **b** have to be expressed in terms of the basis **i**, **j** and **k**

$\mathbf{a} = 4\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}$

$\mathbf{b} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\mathbf{a} \text{ x } \mathbf{b} = ((-5) \times 3 - (-2) \times (-6))\mathbf{i} - (4 \times 3 - (-1) \times (-6))\mathbf{j} + (4 \times (-2) - (-1) \times (-5))\mathbf{k}$

$= -27\mathbf{i} - 6\mathbf{j} - 13\mathbf{k}$In matrix form the calculation can be taken from the determinant of

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & -6 \\ -1 & -2 & 3 \end{pmatrix}$$

2. Find **p x q** where $\mathbf{p} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{q} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Answer:

$\mathbf{p} \text{ x } \mathbf{q} = (3 \times (-2) - 2 \times (-1))\mathbf{i} - ((-2) \times (-2) - 4 \times (-1))\mathbf{j} + ((-2) \times 2 - 4 \times 3)\mathbf{k}$

$= -4\mathbf{i} - 8\mathbf{j} - 16\mathbf{k}$

Algebraic vector product exercise

There is a different exercise on the web.

15 min

Q30: Find **a x b** for the following pairs of vectors

a) $-2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j}$
b) $-5\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
c) $-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

The following table shows the outcome of the vector products of the standard vectors

i x j = **k**	**j x i** = **-k**	**i x i** = 0
j x k = **i**	**k x j** = **-i**	**j x j** = 0
k x i = **j**	**i x k** = **-j**	**k x k** = 0

Properties of vector products

The vector product obeys the distributive laws.

Property 1
a x (**b + c**) = (**a x b**) + (**a x c**) and (**a + b**) **x c** = (**a x c**) + (**b x c**)

Q31: Show that the distributive property **a x** (**b + c**) = (**a x b**) + (**a x c**) holds for the vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - 3\mathbf{k}$

However the vector product does not obey all the laws that the product of real numbers does. For example, from the table of vector products of standard basis vectors shown previously

i x j = k = -j x i

Hence **a x b** need not equal **b x a**

In fact

Property 2
For any two vectors **p** and **qp x q** = **-q x p**

The proof is shown here rather than in the section headed Proofs as it is important.

Let $\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{q} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$

$\mathbf{q} \times \mathbf{p} = (ec - bf)\mathbf{i} - (dc - af)\mathbf{j} + (db - ae)\mathbf{k}$

but $\mathbf{p} \times \mathbf{q} = (bf - ec)\mathbf{i} - (af - dc)\mathbf{j} + (ae - db)\mathbf{k}$

$= -[(ec - bf)\mathbf{i} - (dc - af)\mathbf{j} + (db - ae)\mathbf{k}]$

So $\mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}$

Example Show that $(6\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (9\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = -(9\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \times (6\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ by expanding the brackets and using the properties of the vector product of standard bases.

Answer:

$(6\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (9\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

$= 54\mathbf{i} \times \mathbf{i} - 9\mathbf{j} \times \mathbf{i} + 27\mathbf{k} \times \mathbf{i} + 6\mathbf{i} \times \mathbf{j} - \mathbf{j} \times \mathbf{j} + 3\mathbf{k} \times \mathbf{j} + 24\mathbf{i} \times \mathbf{k} - 4\mathbf{j} \times \mathbf{k} + 12\mathbf{k} \times \mathbf{k}$

$= 9\mathbf{k} + 27\mathbf{j} + 6\mathbf{k} - 3\mathbf{i} - 24\mathbf{j} - 4\mathbf{i}$

$= -7\mathbf{i} + 3\mathbf{j} + 15\mathbf{k}$

and $-(9\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \times (6\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

$= -(54\mathbf{i} \times \mathbf{i} + 6\mathbf{j} \times \mathbf{i} + 24\mathbf{k} \times \mathbf{i} - 9\mathbf{i} \times \mathbf{j} - \mathbf{j} \times \mathbf{j} - 4\mathbf{k} \times \mathbf{j} + 27\mathbf{i} \times \mathbf{k} + 3\mathbf{j} \times \mathbf{k} + 12\mathbf{k} \times \mathbf{k})$

$= -(7\mathbf{i} - 3\mathbf{j} - 15\mathbf{k})$

$= -7\mathbf{i} + 3\mathbf{j} + 15\mathbf{k}$ as required.

From the result $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, then $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) = 0$

If in particular $\mathbf{a} = \mathbf{b}$ then $2(\mathbf{a} \times \mathbf{a}) = 0$ then

Property 3
For any vector $\mathbf{a}$ $\mathbf{a} \times \mathbf{a} = 0$

Another law that holds for the product of real numbers is a(bc) = (ab)c. It does not hold for the vector product.

Property 4
For any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, it is generally the case that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

Example Show that the property $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is true for the vectors $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$

Answer:

$\mathbf{b} \times \mathbf{c} = 3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -13\mathbf{i} - 3\mathbf{j} + 17\mathbf{k}$

$\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -15\mathbf{i} + 15\mathbf{k}$

They are not the same.

Vector product properties exercise

10 min

There is another exercise on the web for you to try if you prefer it.

Q32: Show that **a x a** = 0 for **a** = -3**i** - **j** - **k**

Q33: Show that **p x q** = **-q x p** for the vectors **p** = 2**i** + 3**j** + 4**k** and **q** = 4**i** + **j** + 5**k**

Q34: Show that the distributive law (**a + b**)**x c** = (**a x c**) + (**b x c**) holds for

$\mathbf{a} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

1.6 Vector product - geometric form

Learning Objective

Calculate the vector product using the geometric formula

The second way of expressing a vector product is geometrically by specifying its direction and magnitude.

As shown earlier, the angle between vectors can be found using the scalar product.

It was shown that if **a** • **b** = 0 then the vectors **a** and **b** are perpendicular. This can be used to explore the direction of **a x b**

Example Show that **a** • (**a x b**) = 0 and that **b** • (**a x b**) = 0

Answer:

Let **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k**

Then **a x b** = $(a_2b_3 - b_2a_3)$**i** - $(a_1b_3 - b_1a_3)$**j** + $(a_1b_2 - b_1a_2)$**k**

Thus **a** • (**a x b**) = $a_1a_2b_3 - a_1b_2a_3 - a_2a_1b_3 + a_2b_1a_3 + a_3a_1b_2 - a_3b_1a_2 = 0$

Also **b** • (**a x b**) = $b_1a_2b_3 - b_1b_2a_3 - b_2a_1b_3 + b_2b_1a_3 + b_3a_1b_2 - b_3b_1a_2 = 0$

Since **a** • (**a x b**) = 0 and **b** • (**a x b**) = 0 the vector **a x b** is perpendicular to both **a** and **b**

But **a x b** = **-b x a** from the properties of vector products given earlier. So **a x b** gives a vector perpendicular to **a** and **b** in a positive direction whereas **b x a** also gives a vector perpendicular to **a** and **b** but in the negative and opposite direction.

This direction can be determined by **the right hand rule** which states that if the right hand is used with the thumb pointing in the direction of **a** and the first finger pointing in the direction of **b**, then the middle finger points in the direction of **a x b**

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The vector **a x b** has length $|$ **a** $||$ **b** $|$ sin θ (see Proof 2 in the Proof section).

Thus the direction and length of the vector **a x b** have been specified.

Vector product in geometric form

The vector product of **a** and **b** is defined with

- magnitude of $|$ **a x b** $|$ = $|$ **a** $||$ **b** $|$ sin θ, where θ is the angle between **a** and **b**, $0 \leq \theta \leq 180°$
- direction perpendicular to both vectors **a** and **b** as determined by the right hand rule.

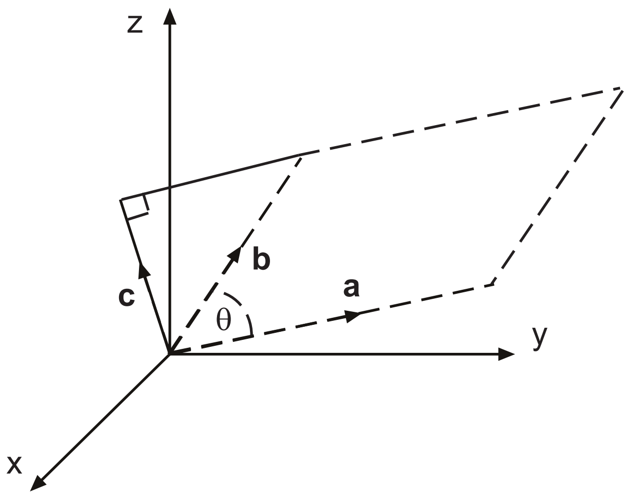
Note that **a x b** = **0** if and only if **a** and **b** are parallel.

Example Find the magnitude of the vector product of the two vectors which are at an angle of 50 ° to each other and have lengths of 5 and 7 units

Answer:

$|$ **a x b** $|$ = 5 $\times$ 7 $\times$ sin 50° = 26.81

The length of the vector product **a x b** has the same formula as the area of a parallelogram in which the vectors **a** and **b** are the sides of the parallelogram as shown in the diagram.



Area of parallelogram = $|$ **a** $||$ **c** $|$

But $|$ **c** $|$ = $|$ **b** $|$ sin θ

Thus the area = $|$ **a** $||$ **b** $|$ sin θ = $|$ **a x b** $|$

Of course to find **a x b** the algebraic version of the vector product is used.

Example Find the area of the parallelogram which has adjacent edges

a = 3**i** + **j** + 2**k** and **b** = 2**i** - **j**

Answer:

a x b = 2**i** + 4**j** - 5**k**

Hence | **a x b** | = $\sqrt{2^2 + 4^2 + (-5)^2}$ = 6.708 units.

The area of the parallelogram is 6.708 sq units.

Geometric vector product exercise

There is a web exercise for you to try if you prefer it.

15 min

Q35: Find a vector perpendicular to 2**i** - **j** + 3**k** and **i** + 2**j**

Q36: Find the length of **a x b** given that **a** has length 4 units, **b** has length 5 units and the angle between them is 45°. Give the answer to 2 decimal places.

Q37: Find the area of a parallelogram with edges **i** - 4**j** + **k** and 2**i** + 3**j** - 2**k**

Give the answer to 3 decimal places.

Q38: Find the area of a triangle with vertices at (0, 0, 0), (3, 1, 2), (2, -1, 0). Give your answer to 3 decimal places.

Q39: Find the area of the triangle with vertices (1, -1, 0), (2, 1, -1) and (-1, 1, 2)

Activity

Investigate geometrically **(a x b) x c** and **a x (b x c)** by choosing some easy points to manipulate in three-dimensional space.

1.7 Scalar triple product

Learning Objective

Calculate the scalar triple product of vectors.

To show that **a x b** is perpendicular to **a** and to **b**, the numbers **a** • (**a x b**) and **b** • (**a x b**) are found. If **a**, **b** and **c** are vectors then as **b x c** is also a vector it is possible to find the scalar product of **a** with **b x c**

Scalar triple product

The scalar triple product **a • (b x c)** is the number given by the scalar product of the two vectors **a** and (**b x c**)

This product can also be found using the determinant of a matrix in a similar way to the vector product. Suppose that

a = a_1**i** + a_2**j** + a_3**k**

b = b_1**i** + b_2**j** + b_3**k**

c = c_1**i** + c_2**j** + c_3**k**

This is the determinant of the matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

The calculation of **a • (b x c)** gives $a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$

It is sometimes known as the triple scalar product or the dot cross product.

The calculation is best shown by example.

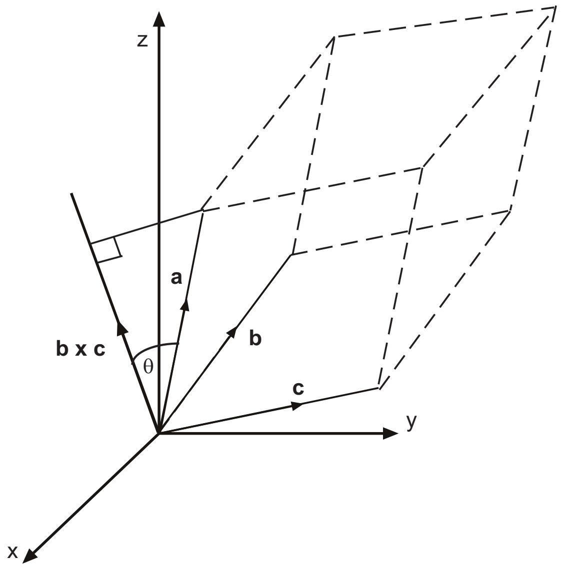
Example Find **a • (b x c)** when **a** = **i** + 2**j** - 3**k**, **b** = 4**i** - **j** + 2**k** and **c** = 3**i** - 2**j** + 5**k**

Answer:

b x c = -**i** - 14**j** - 5**k**

a • (b x c) = -1 -28 +15 = -14

The modulus of the scalar triple product is the volume of a parallelepiped with adjacent edges **a**, **b** and **c** as shown in the following diagram.



The base is formed by **b** and **c**

The area of the base = | **b x c** |

The perpendicular height is | **a** | cos θ

But the volume of the parallelepiped = area of base $\times$ perpendicular height

Thus the volume = | **b x c** | $\times$ | **a** | cos θ = | **a** • (**b x c**) |

(Remember that **a • b** = **b •a**)

Example Find the volume of the parallelepiped with edges **i** + **j** + 2**k**, **i** + 2**j** + 3**k** and 3**i** - **j** - **k**

Answer:

This is the modulus of the scalar triple product.

Let **a** = **i** + **j** + 2**k**, **b** = **i** + 2**j** + 3**k** and **c** = 3**i** - **j** - **k**

The volume is $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$

$\mathbf{b} \times \mathbf{c} = \mathbf{i} + 10\mathbf{j} - 7\mathbf{k}$

Then $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (1 \times 1) + (1 \times 10) + (2 \times (-7)) = 1 + 10 - 14 = -3$

The modulus is 3 and the volume of the parallelepiped is 3 cubic units.

Note that the volume of a parallelepiped is zero if and only if **a**, **b** and **c** are coplanar, that is if $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})| = 0$

If **b** and **c** form the base of the parallelepiped then **b x c** is perpendicular to the base. However, if **a** lies in the base formed by **b** and **c** then **a** is perpendicular to **b x c** which means that the scalar product of **a** and **b x c** is zero.

Scalar triple product exercise

There is another exercise on the web for you to try if you prefer it.

15 min

Q40: Find $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ if $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

Q41: Find $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ if $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 4\mathbf{i} + \mathbf{k}$

Q42: Find the volume of the parallelepiped with edges

$-\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$, $3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $4\mathbf{i} + 2\mathbf{k}$

Q43: Show that the position vectors to the points (1, 2, 3), (4, -1, 2) and (2, -5, -4) lie in the same plane.

Properties of the scalar triple product

The volume of a parallelepiped was shown as $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$. By considering the base of the parallelogram formed from **a** and **b**, the volume is similarly computed as $|\mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})|$ and by taking the base formed from **a** and **c** the volume is $|\mathbf{b} \bullet (\mathbf{c} \times \mathbf{a})|$

It is possible that $\mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})$ and $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ may have different signs, but provided that **a** is followed by **b** which in turn is followed by **c** and back to **a** they have the same signs.

Thus

Property 1
$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})$

The proof of this is shown in the section called Proofs at Proof 8.

If the order is different then the answer need not be the same. For example:

$\mathbf{a} \bullet (\mathbf{c} \times \mathbf{b})$

$= \mathbf{a} \bullet (-(\mathbf{b} \times \mathbf{c}))$ (using the property of vector products)

$= -\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ (using the property of scalar products)

Q44: Check that this property holds for the vectors

$\mathbf{a} = \mathbf{i} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

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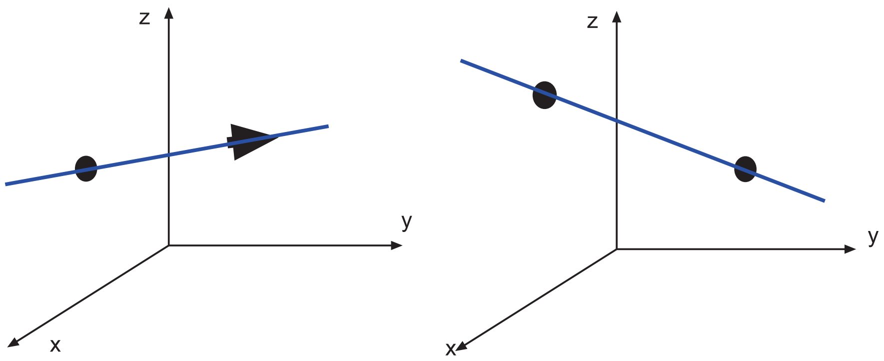
1.8 Lines in three-dimensional space

Learning Objective

Determine and use the equations of a line in vector, parametric and symmetric form

In a similar way to lines in the plane, lines in space are specified by either:

- One point and a direction.
- Two points on the line.



1.8.1 The equation of a line

There are three forms in which the equation of a line can be written in three-dimensional space. These are vector, parametric and symmetric form.

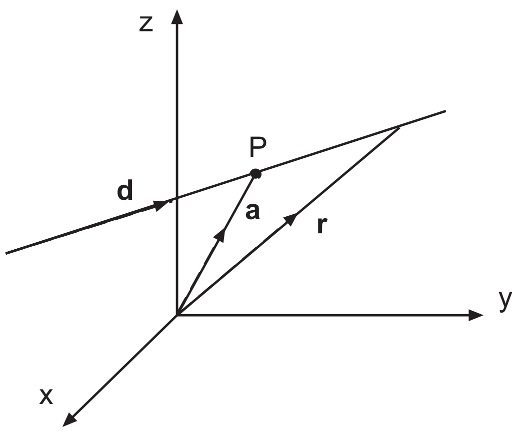
Suppose first that the line is specified by a point and a direction.

Vector form of the equation of a line

The vector equation of a line through the point P with direction **d** is

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where $\mathbf{a} = \overrightarrow{OP}$, **d** is a vector parallel to the required line and λ is a real number.

Any position vector **r** of a point on the required line will satisfy this equation.



Example Find the vector equation of the straight line through (2, -1, 6) and parallel to the vector **i** + 2**j** - 8**k**

Answer:

Using the formula $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{d}$ with $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ and $\mathbf{d} = \mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$ gives

$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 8\mathbf{k})$

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ then the equation can be split into component parts. This leads to the second form of the equation of a straight line.

Parametric form of the equation of a line

The parametric equations of a line through the point $P = (a_1, a_2, a_3)$ with direction $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ are

$x = a_1 + \lambda d_1$, $y = a_2 + \lambda d_2$, $z = a_3 + \lambda d_3$ where λ is a real number.

Example Find the parametric equations of the line through (3, 4, 5) and parallel to the vector $2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

Answer:

Using the formulae with $a_1 = 3$, $a_2 = 4$ and $a_3 = 5$

$x = 3 + 2\lambda$, $y = 4 - 3\lambda$, $z = 5 - 4\lambda$

It is easy to obtain the parametric form from the vector form and vice versa.

Examples

1. State the parametric form of the equation of the line which has a vector equation of $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where P is the point (3, -1, 5), $\mathbf{a} = \overrightarrow{OP}$ and $\mathbf{d} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

Answer:

$\mathbf{r} = (3\mathbf{i} - \mathbf{j} + 5\mathbf{k}) + \lambda(-2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

Equating each component in turn gives

$x = 3 - 2\lambda$, $y = -1 + 4\lambda$, $z = 5 - \lambda$

2. Find the vector form of the equation of a line with parametric equations

$x = -1 + 3\lambda$, $y = \lambda$, $z = 2 + 2\lambda$

Answer:

$x = -1 + 3\lambda$, $y = \lambda$, $z = 2 + 2\lambda$ gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$\mathbf{a} = -\mathbf{i} + 2\mathbf{k}$ and $\mathbf{d} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

The equation is $\mathbf{r} = (-\mathbf{i} + 2\mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Equations of the parametric form can be rearranged to eliminate the real number λ giving the third form of the equation of a straight line.

Symmetric form of the equation of a line

If $x = a_1 + \lambda d_1$, $y = a_2 + \lambda d_2$, $z = a_3 + \lambda d_3$ are parametric equations of a line, the symmetric equation of this line is

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

These symmetric equations of a line are also known as the Cartesian equations of a line.

Note that if any of the denominators are zero then the corresponding numerator is also zero. This means that the vector is parallel to an axis.

It is also important to ensure that the symmetric form of the equation of a line is stated with each coefficient of x, y and z on the numerator equal to 1. An equation with any other coefficients for x, y and z is not the symmetric equation of the line.

The direction ratios are the denominators d_1, d_2 and d_3 in these symmetric equations.

It is straightforward to obtain the symmetric form from either the parametric or vector form. The following examples will demonstrate how this is done.

Example Find the parametric and symmetric equations of the line with a vector equation of

$\mathbf{r} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + \lambda(-\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Answer:

$a = (2, -3, 1)$ and $\mathbf{d} = (-\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

The parametric equations are $x = 2 - \lambda$, $y = -3 - 2\lambda$ and $z = 1 + \lambda$

Solving each component for λ gives the symmetric equations as

$$\frac{x - 2}{-1} = \frac{y + 3}{-2} = \frac{z - 1}{1}$$

Note that even if the denominator is 1, the form of equations requires that it should be left there.

Examples

1. Find the parametric and symmetric equations of the line which has position vector $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and is parallel to the vector $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Answer:

The parametric equations are $x = 1 + 3\lambda$, $y = -1 + 2\lambda$, $z = 1 + \lambda$

and so

$$\frac{x - 1}{3} = \frac{y + 1}{2} = \frac{z - 1}{1} = \lambda$$

Hence by eliminating λ the symmetric equations are

$$\frac{x - 1}{3} = \frac{y + 1}{2} = \frac{z - 1}{1}$$

2. Find the symmetric equations of the line which has parametric equations

$x = 2 - 2\lambda$, $y = -1 + 3\lambda$, $z = -\lambda$

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Answer:

Rearrange to make each equation equal to λ

This gives $\frac{x-2}{-2} = \frac{y+1}{3} = \frac{z-0}{-1} = \lambda$

Hence the symmetric equations are $\frac{x-2}{-2} = \frac{y+1}{3} = \frac{z-0}{-1}$

To convert the symmetric equations of a line into the parametric form, take the equation $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$ and call this common value the parameter λ

Then solve for x, y and z in terms of λ

Examples

1. Find the parametric form of the equation of a line with symmetric equations $\frac{x+3}{2} = \frac{y-1}{1} = \frac{z-4}{-3}$

Answer:

Make each part of the equation equal to λ and solve for x, y and z

$\frac{x+3}{2} = \lambda$ so $x = -3 + 2\lambda$

$\frac{y-1}{1} = \lambda$ so $y = 1 + \lambda$

$\frac{z-4}{-3} = \lambda$ so $z = 4 - 3\lambda$

The equations are $x = -3 + 2\lambda$, $y = 1 + \lambda$, $z = 4 - 3\lambda$

2. Find the parametric and vector forms of the equation of a line which has symmetric equations $\frac{x+4}{1} = \frac{y-3}{-2} = \frac{z-4}{2}$

Answer:

Let $\frac{x+4}{1} = \frac{y-3}{-2} = \frac{z-4}{2} = \lambda$

Solving for x, y and z gives: $x = -4 + \lambda$, $y = 3 - 2\lambda$, $z = 4 + 2\lambda$

This gives the parametric equations of the line.

Thus $\mathbf{a} = -4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

Thus $\mathbf{r} = (-4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ is the vector equation of the line.

Equations of a line are not unique. The direction vectors however, are proportional (one is a scalar multiple of the other).

Example Suppose that the equation of a line is given by the symmetric equations $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-1}{1}$

then in parametric form the equations are

$x = 3 + 2\lambda$, $y = -1 + 3\lambda$, $z = 1 + \lambda$

Answer:

If $\lambda = 2$, a point on this line is = (7, 5, 3)

Consider the line with symmetric equations $\frac{x - 9}{-4} = \frac{y - 8}{-6} = \frac{z - 4}{-2}$

This has symmetric equations x = 9 - 4λ, y = 8 - 6λ, z = 4 - 2λ

Substitution of the x, y, and z values of the point (7, 5, 3) into either of the forms of this second line gives $\lambda = {}^1/_2$ and so the point is also on this line.

But the direction vector $\mathbf{d}_1 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ of the first line is a scalar multiple of the direction vector $\mathbf{d}_2 = -4\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ of the second line.

Thus the lines are in fact the same line with equations that look very different.

10 min

Equation of a line exercise

There is another version of this exercise on the web for you to try if you wish.

Q45: State the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the line -2**i** - **j** + 3**k**

Q46: Find the vector equation of a line with parametric equations:

x = 3 + 4λ, y = -2 - 3λ and z = 4 - λ

Q47: Give the parametric and symmetric equations for the lines:

a) through (-1, 3, 0) in the direction -2**i** - **j** + 4**k**
b) through (2, -1, 0) in the direction -**i** + 4**k**

Q48: Find the symmetric equations of the lines with vector equations:

a) $\mathbf{r} = -\mathbf{j} + 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$
b) $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

Q49: From the vector equation of the line given as $(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + \lambda\,(3\mathbf{i} - \mathbf{j} - \mathbf{k})$ state the point with y coordinate -1 which also lies on this line.

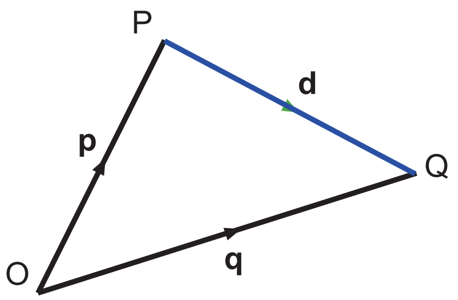
Q50: Find the parametric equations of the line through the point (3, 4, 5) and parallel to the line **i** + **j** + **k**

Q51: Find the symmetric equation of the line which has parametric equations of

x = - λ, y = - 2 + λ, z = 3 + 2λ

Line determined by two points

Now suppose that the line is determined by two points P and Q. A direction vector on this line can be obtained by taking $\overrightarrow{PQ}$



Let P be the point (a_1, a_2, a_3) and Q be the point (b_1, b_2, b_3) then a direction vector is represented by the line segment $\overrightarrow{PQ}$ = **d** = **q** - **p**. It is now possible to obtain vector, parametric and symmetric equations of the line as before.

Example Find the vector, parametric and symmetric equations of the line through (1, -2, -1) and (2, 3, 1)

Answer:

If **a** = the position vector **i** - 2**j** - **k** and **b** = 2**i** + 3**j** + **k**

then a direction vector is **b - a** = **i** + 5**j** + 2**k**

The vector form of the equation is **r** = **i** - 2**j** - **k** + λ(**i** + 5**j** + 2**k**)

The parametric equations are x = 1 + λ, y = -2 + 5λ, z = -1 + 2λ

The symmetric equations are $\frac{x - 1}{1} = \frac{y + 2}{5} = \frac{z + 1}{2}$

Equation from two points exercise

There is another exercise on the web for you to try if you prefer it.

15 min

Q52: Give parametric and symmetric equations for the line through the points (3, -1, 6) and (0, -3, -1)

Q53: Find the parametric and symmetric equations of the line joining the points A (1, 2, -1) and B(-1, 0, 1)

1.8.2 Intersection of and the angles between lines

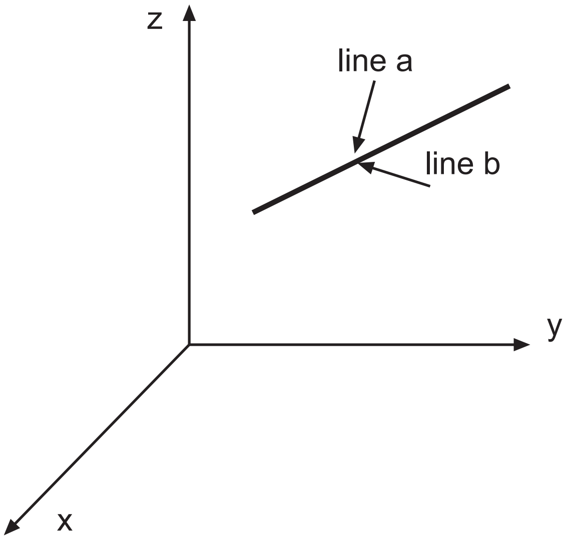
Learning Objective

Find the intersection of and the angle between two lines

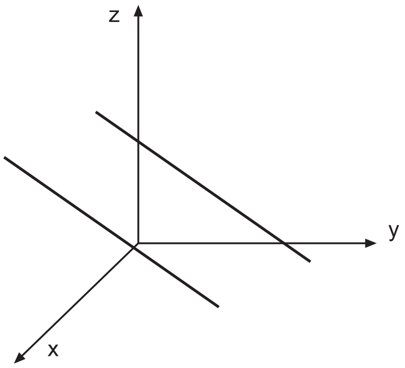
Intersection of two lines in three-dimensional space

Now suppose that L_1 and L_2 are two lines in three-dimensional space. The two lines can:

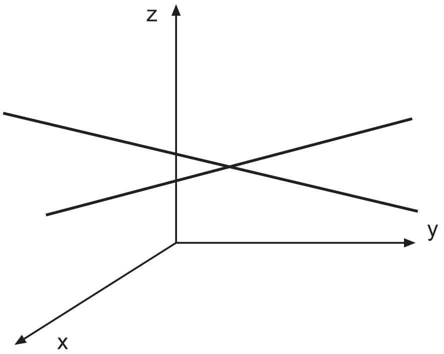
- be identical



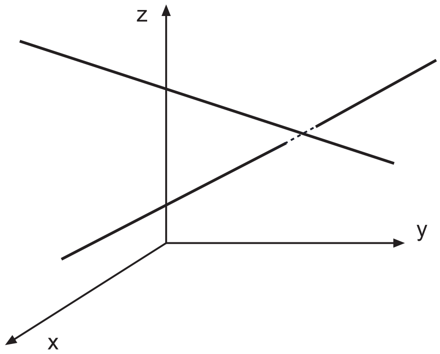
- be parallel



- intersect



- be skew, that is, not identical, parallel or intersecting



The last case cannot arise in two-dimensional space.

Note that if two lines are parallel then their direction ratios are proportional.

Two lines intersecting at a point

In three dimensions lines normally do not intersect. If however they do meet then by equating the parametric equations, there will be a unique solution. To show this, take the lines in parametric form.

Example Suppose that L_1 has parametric equations x = 1 + λ, y = -1 + λ, z = 2 + λ

and L_2 has parametric equations x = 2 + μ, y = -2 + 3μ, z = -1 + 5μ

Show that the two lines L_1 and L_2 intersect but are not the same line.

Answer:

As their direction vectors are not proportional L_1 and L_2 are not the same line.

If the lines intersect then there are solutions to the set of equations obtained by equating, in turn, x, y and z

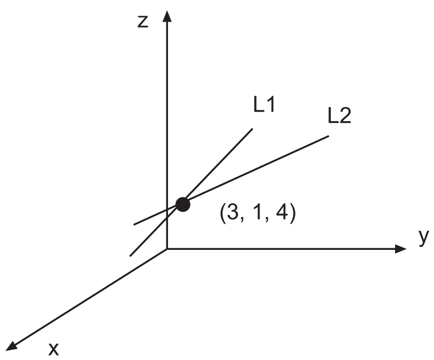
1 + λ = 2 + μ (this is the x equations).

-1 + λ = -2 + 3μ (this is the y equations).

2 + λ = -1 + 5μ (this is the z equations).

These can be solved to give μ = 1 and λ = 2

Since there is a solution, the lines meet. By substituting the values for μ and λ the point of intersection is (3, 1, 4)



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Two skew lines

If the equations of two lines have no solution for the real numbers λ and μ the lines are skew.

Example Show that the lines L_1 with symmetric equation $\frac{x - 4}{3} = \frac{y - 0}{-1} = \frac{z - 2}{-1}$ and L_2 with parametric equations x = μ, y = μ, z = 3 + μ do not intersect.

Answer:

If L_1 and L_2 do intersect then there is a solution of the set of equations

1) $4 + 3\lambda = \mu$

2) $-\lambda = \mu$

3) $2 - \lambda = 3 + \mu$

However since $4 + 3\lambda = \mu = -\lambda$

then $\lambda = -1$

Substituting $\lambda = -1$ in equation 2) gives $\mu = 1$ but this is inconsistent with equation 3) which, using these values of λ and μ gives 3 = 4

There is no solution to these equations and so the lines do not intersect.

Angle between two lines

If two lines intersect then the angle between them is the angle between their direction vectors. The acute angle is always taken.

Example Show that the two lines L_1 and L_2, which have symmetric equations as shown, intersect and find the angle between them.

L_1: $\frac{x + 4}{2} = \frac{y - 5}{-4} = \frac{z - 3}{1}$

L_2: $\frac{x - 0}{-1} = \frac{y - 3}{-1} = \frac{z - 2}{1}$

Answer:

First of all find the parametric equations that are

L_1 : $x = -4 + 2\lambda$, $y = 5 - 4\lambda$, $z = 3 + \lambda$

L_2 : $x = -\mu$, $y = 3 - \mu$, $z = 2 + \mu$

The set of equations are

$-4 + 2\lambda = -\mu$

$5 - 4\lambda = 3 - \mu$

$3 + \lambda = 2 + \mu$

These can be solved to give $\lambda = 1$ and $\mu = 2$

The two lines meet at (-2, 1, 4)

The direction vectors are $\mathbf{d_1} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ for L_1 and $\mathbf{d_2} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ for L_2

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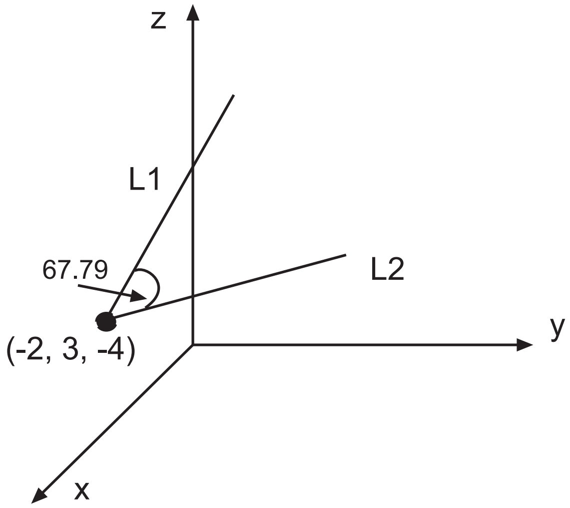
If θ is the angle between L_1 and L_2 then

$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|} = \frac{3}{\sqrt{21}\sqrt{3}}$

So the angle is 67.79°



Note: If two lines **a** and **b** have direction cosines equal to a_1, a_2, a_3 and b_1, b_2, b_3 then the angle can be found from the equation $\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

Suppose that each direction cosine could be expressed as

$\cos\alpha = \frac{a}{|\mathbf{a}|} = a_1$

$\cos\beta = \frac{b}{|\mathbf{a}|} = a_2$

$\cos\gamma = \frac{c}{|\mathbf{a}|} = a_3$ for point (a, b, c)

and similarly

$\cos\alpha = \frac{d}{|\mathbf{b}|} = b_1$

$\cos\beta = \frac{e}{|\mathbf{b}|} = b_2$

$\cos\gamma = \frac{f}{|\mathbf{b}|} = b_3$ for point (d, e, f)

then $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

so $\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{ad + be + cf}{|\mathbf{a}||\mathbf{b}|} = a_1b_1 + a_2b_2 + a_3b_3$

Intersection points and angles exercise

15 min

There is another mixed exercise on the web for you to try if you like.

Q54: Show that the lines $\frac{x-2}{1} = \frac{y-2}{3} = \frac{z-3}{1}$ and $\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-4}{2}$ intersect.

Q55: Determine whether the two lines L_1 and L_2 intersect when

L_1 has symmetric equations $\frac{x+3}{1} = \frac{y-2}{1} = \frac{z-1}{-3}$ and L_2 has symmetric equations $\frac{x+4}{-2} = \frac{y-1}{1} = \frac{z-0}{1}$

Q56: Find the intersection of and the angle between the lines L_1 with parametric equations $x = -2 + 2\lambda$, $y = 1 - 3\lambda$, $z = -1 + \lambda$ and the line L_2 which passes through the point (-3, 4, 0) and is parallel to **-i** + **j** - **k**

Q57: Find the angle between and the point of intersection of the two lines

$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

Q58: Find the point of intersection of the line through the point (1, 3, -2) and parallel to 4**i** + **k** with the line through the point (5, 3, 8) and parallel to **-i** + 2**k**

Then find the angle between them.

Q59: Find the intersection of and the angle between the lines L_1 with symmetric equations $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-2}{1}$ and L_2 with symmetric equations $\frac{x+1}{2} = \frac{y-6}{1} = \frac{z-7}{-1}$

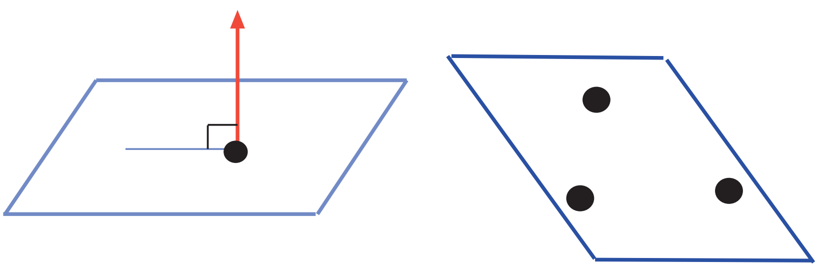
1.9 Planes in three-dimensional space

Learning Objective

Determine and use equations of the plane

Planes in space are specified either by:

- One point on the plane and a direction **n** at right angles (normal) to the plane.
- Three non-collinear points in the plane.



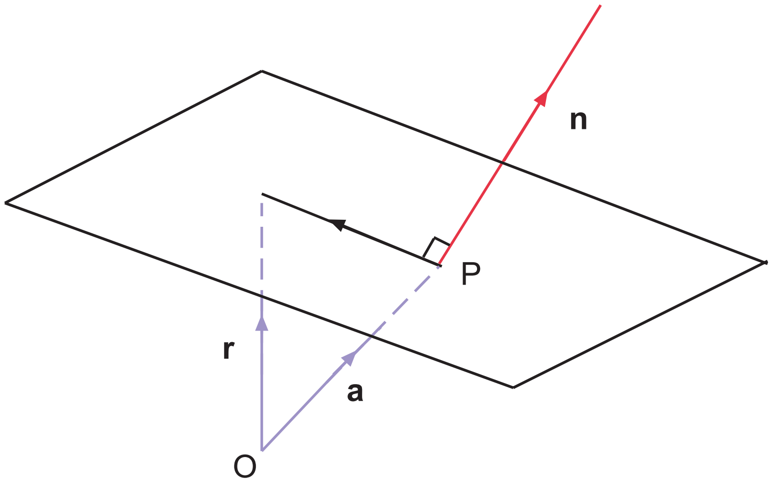
1.9.1 The equation of a plane

There are three common forms in which the equation of a plane can be written. These are the vector equation, the Cartesian equation and the parametric equation.

Vector equation of a plane

The vector equation of a plane containing the point P and perpendicular to the vector **n** is **(r - a) •n** = 0 where **a** = $\overrightarrow{OP}$

The equation can also be written as **r • n** = **a • n**



Example Find the vector equation of the plane through (-1, 2, 1) with normal vector **i** - 3**j** + 2**k**

Answer:

The general equation is **(r - a) • n** = 0 or **r • n** = **a •n**

Here **a** = -**i** + 2**j** + **k**

and **n** = **i** - 3**j** + 2**k**

The vector equation is

r • (**i** - 3**j** + 2**k**) = -1 - 6 + 2 = -5

If **r** = x**i** + y**j** + z**k** then evaluating the scalar products **r • n** and **a • n** will give the second form of the equation of a line.

This is called the Cartesian form of the equation of the plane.

Cartesian equation of a plane

The Cartesian equation of a plane containing the point P (a_1, a_2, a_3) and perpendicular to the vector **n** = n_1**i** + n_2**j** + n_3**k** is $xn_1 + yn_2 + zn_3 = d$ where $d = a_1n_1 + a_2n_2 + a_3n_3$

Example Find the Cartesian equation of the plane through (-1, 2, 1) which has normal vector **i** - 3**j** + 2**k**

Answer:

The vector form of this plane given in the previous example is **r •** (**i** - 3**j** + 2**k**) = -5

If **r** = x**i** + y**j** + z**k** then evaluating the vector product **r • n** gives x - 3y + 2z = -5

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Alternatively, the Cartesian equation in the last example can be found by immediate substitution into the definition of the equation in this form.

The Cartesian form of the equation of a plane is the simplification of the vector equation.

It is straightforward to find the vector equation of a plane if the Cartesian form is known.

Example Find the vector equation of a plane whose Cartesian equation is
2x - y + 3z = -4

Answer:

If **r** = x**i** + y**j** + z**k** then 2x - y + 3z = **r** •(2**i** - **j** + 3**k**)

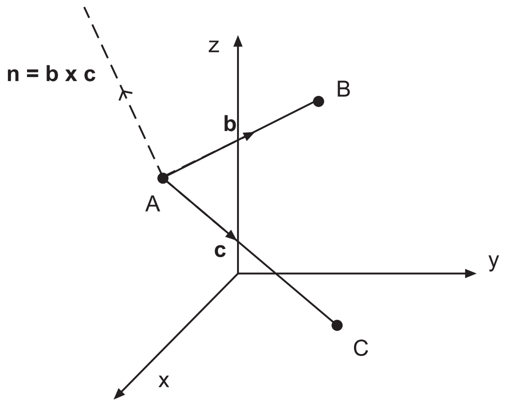
The vector equation is **r** •(2**i** - **j** + 3**k**) = -4

The examples shown previously have defined the plane using a point on the plane and a normal to the plane. It is also possible to define the plane from three non-collinear points.

The vector equation of a plane from three non-collinear points

Suppose that A, B and C are three non-collinear points. The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in a plane through all three points.

Their vector product $\overrightarrow{AB}$ **x** $\overrightarrow{AC}$ is perpendicular to this plane. Any multiple of the vector product forms a normal **n** to the plane. Since **a** is the position vector to the point A that lies in the plane it is straightforward to give the equation of the plane as **r** • **n** = **a** • **n**



Example Find the equation of the plane through A(-1, 3, 1), B(1, -3, -3) and C(3, -1, 5)

Answer:

$\overrightarrow{AB}$ = 2**i** - 6**j** - 4**k**

$\overrightarrow{AC}$ = 4**i** - 4**j** + 4**k**

These two vectors lie in the plane and their vector product is normal to the plane.

$\overrightarrow{AB}$ **x** $\overrightarrow{AC}$ = -40**i** - 24**j** + 16**k**

Let **n** = -5**i** - 3**j** + 2**k** (**n** is $^1/_8$ of the vector product for ease)

and **a** = -**i** + 3**j** + **k**

Thus the equation of the plane is **r** • **n** = **a** •**n**

So -5x - 3y + 2z = 5 - 9 + 2 = -2

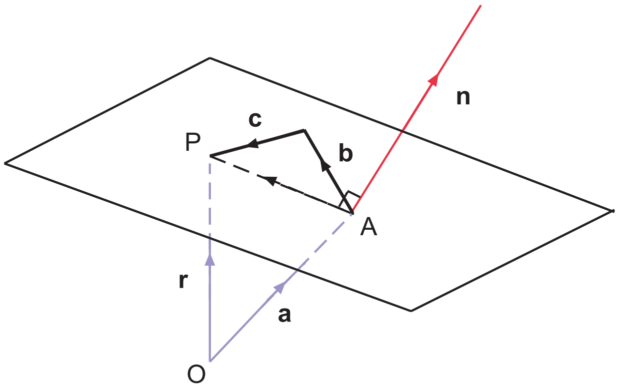
The equation is 5x + 3y - 2z = 2

The third form of the equation of the plane is the parametric equation.

Parametric equation of a plane

The parametric equation of the plane is

r = **a** + λ**b** + μ**c** where **a** is a position vector of a point in the plane, **b** and **c** are two non-parallel vectors parallel to the plane and λ and μ are real numbers.



Note that the parametric equation of the plane is sometimes referred to as the symmetric equation of the plane. Do not confuse this with the distinct differences in the equations of a line in parametric and symmetric form.

Example Find the parametric equation of the plane through (2, 3, -1) and parallel to the vectors

-2**i** - 3**j** + **k** and -**i** + 3**j** + 4**k**

Answer:

The equation is **r** = **a** + λ**b** + μ**c** where

a = 2**i** + 3**j** - **k**, **b** = -2**i** - 3**j** + **k** and **c** = -**i** + 3**j** + 4**k**

The Cartesian form, ax + by + cz = d can be thought of as one linear equation in three unknowns and solved by putting y = λ and z = μ to give the parametric solution.

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Example Find the parametric equation of the plane with Cartesian equation $x + 2y + z = 3$

Answer:

To obtain the parametric equation let $y = \lambda$ and $z = \mu$, then $x = 3 - 2\lambda + \mu$

so the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Let $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Then the parametric equation of the plane is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

It is also possible to find the Cartesian form of the equation of a plane from the parametric form of $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

By definition of the parametric form, **b** and **c** are vectors parallel to the plane.

Thus **b x c** is a normal to the plane.

Evaluating $\mathbf{r} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{r} \bullet \mathbf{a}$ gives the Cartesian form of the equation of the plane.

Example Find the Cartesian form of the equation of a plane given the parametric form of

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ where $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = 2\mathbf{j} - \mathbf{k}$

Answer:

$\mathbf{b} \times \mathbf{c} = -5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

This is a normal **n** and **a** is the position vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

The equation of the plane is $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$

So $-5x + y + 2z = -5 + 2 - 2 = -5$

It is worth noting that two parallel or coincident planes have normals that are proportional.

Equation of a plane exercise

15 min

There is an exercise on the web for you to try if you like.

Q60: Find the equation of the plane which passes through (1, 2, 3) and is perpendicular to $6\mathbf{i} - \mathbf{j} + \frac{1}{3}\mathbf{k}$

Q61: Find the equation of a plane through the points

A = (1, 1, -1), B = (2, 0, 2) and C = (0, -2, 1)

Q62: Find the equation of a plane which passes through the points

A = (1, 2, 1), B = (-1, 0, 3) and C = (0, 5, -1)

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Q63: Find the parametric form of the equation of the plane which has Cartesian equation x - 2y - 6z = -3

1.9.2 Intersection of and the angles between two planes

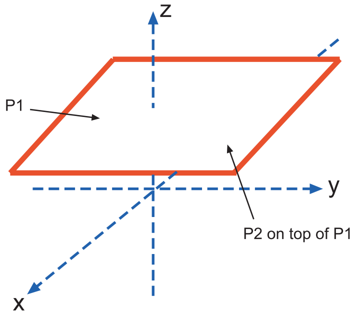
Learning Objective

Find the intersection of and the angle between two planes

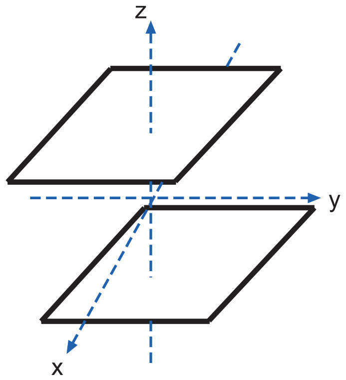
Consider two planes in three-dimensional space.

In relation to each other, the two planes can:

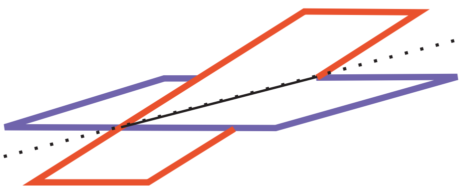
- be coincident



- be parallel



- intersect on a line



Note: parallel or coincident planes have normals that are proportional.

Example : Parallel planes

Determine whether the two planes P_1 and P_2 with equations 6x + 2y - 2z = 5 and -3x - y + z = 2 respectively are parallel, coincident or intersect.

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Answer:

A normal **n** for P_1 is 6**i** + 2**j** - 2**k**

A normal **m** for P_2 is -3**i** - **j** + **k**

As **n** = -2**m**, $P_1$and P_2 are either parallel or coincident.

If P_1 and P_2 did intersect, the two equations 6x + 2y - 2z = 5 and -3x - y + z = 2 would have a common solution for x, y and z.

This set of two equations have no solution and so P_1 and P_2 have no points in common.

Hence, the planes do not intersect and are not coincident.

The planes are parallel.

If two planes intersect, Gaussian elimination can be used to find the vector equation of the line of intersection. The symmetric form of the equation can also be found more directly by algebraic manipulation of the equations. The following examples demonstrate this.

Examples

1. Intersection of two planes in a line by Gaussian elimination

Find the vector equation of the line of intersection of the two planes P_1 with equation 4x + y - 2z = 3 and P_2 with equation x + y - z = 1

Answer:

Using Gaussian elimination on the two equations gives

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 4 & 1 & -2 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -3 & 2 & -1 \end{array}\right)$$

That is, x + y - z = 1 and -3y + 2z = -1

Let z = λ

Then by substituting in -3y + 2z = -1 gives y = $\frac{1}{3} + \frac{2}{3}\lambda$ and

by substituting in x + y - z = 1 gives x = $\frac{2}{3} + \frac{1}{3}\lambda$

The vector equation is **r** = **a** + λ**d** = $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$

2. Intersection of two planes in a line by algebraic manipulation

Find the symmetric equation of the line of intersection of the two planes P_1 with equation 4x + y - 2z = 3 and P_2 with equation x + y - z = 1

Answer:

Subtract $2P_2$ from P_1 to eliminate z and give the equation 2x - y = 1

Subtract P_2 from P_1 to eliminate y and give the equation 3x - z = 2

Solving both for x gives x = $\frac{y+1}{2} = \frac{z+2}{3}$ or as an equation of a line in symmetric form this is $\frac{x-0}{1} = \frac{y+1}{2} = \frac{z+2}{3}$

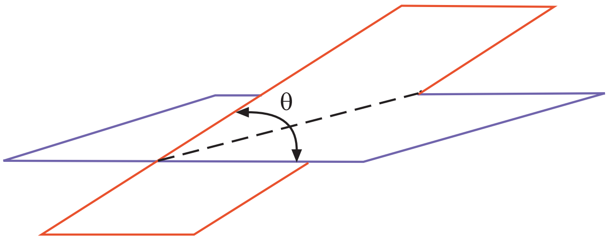
Angle between two planes

If two planes intersect, the angle between them is equal to the angle between their normal vectors. The equation $\cos\theta = \frac{\mathbf{n} \bullet \mathbf{m}}{|\mathbf{n}||\mathbf{m}|}$ will give the angle where **n** and **m** are the normals.

Example Find the angle between the two planes with equations

2x + y - 2z = 5 and 3x - 6y - 2z = 7

Answer:



The normal vectors are **n** = 2**i** + **j** - 2**k** and **m** = 3**i** - 6**j** - 2**k**

$\cos\theta = \frac{4}{|3||7|} = \frac{4}{21}$ so $\theta = 79°$

Intersection and angles exercise

30 min

There is another exercise on the web if you want to try something different.

Q64: Find the equation of the plane that passes through (1, -3, 2) and is parallel to x - 2y + z = 5

Q65: Ascertain whether the two planes 3x - 6y +9 z = -12 and -x + 2y - 3z = 10 intersect and give the equation if they do.

Q66: Find the equation of a plane in Cartesian form which passes through the point (1, -1, 3) and is parallel to the plane 3x + y + z = 7

Q67: Find the symmetric equation of the line of intersection of the two planes 3x + y - z = -3 and x - y - z = 1

Q68: Ascertain whether the two planes 3x - y + 2z = 5 and x + 2y - z = 4 intersect and if they do give the equation of the line in parametric form.

Q69: Find the angle between the planes x + y - 4z = -1 and 2x - 3y + 4z = -5

Q70: Find the acute angle between the planes 2x - y = 0 and x + y + z = 0

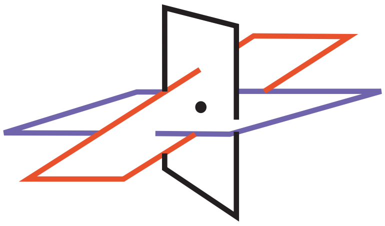
1.9.3 Intersection of three planes

Learning Objective

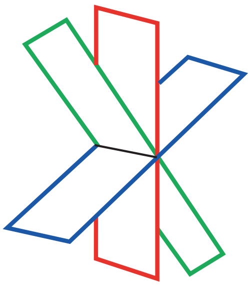
Find the intersection of and the angle between three planes

Consider three distinct non-parallel planes in three-dimensional space. If no two of the three planes are parallel or coincident, the three planes can:

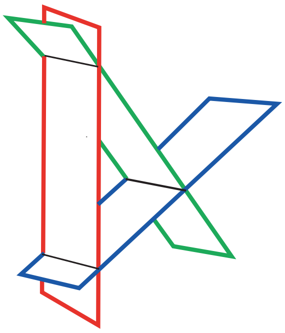
- intersect at a point



- intersect at a line



- not intersect



The technique of Gaussian elimination can be used to find the point or line of intersection of three planes.

Examples

1. Intersection of three planes at a point

Find the point of intersection between the three planes

P_1 : x - 2y - z = 3

P_2 : 2x + y - z = -1

P_3 : -x + y + 2z = 2

Answer:

Gaussian elimination gives

$$\begin{pmatrix} 1 & -2 & -1 & 3 \\ 2 & 1 & -1 & -1 \\ -1 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 & 3 \\ 0 & 5 & 1 & -7 \\ 0 & -1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 & 3 \\ 0 & 5 & 1 & -7 \\ 0 & 0 & 6 & 18 \end{pmatrix}$$

This gives z = 3 and substituting this value back into row 2 gives 5y + 3 = -7

Thus y = -2

A further substitution gives x = 2 and the full solution of x = 2, y = -2 and z = 3

The point of intersection is (2, -2, 3)

2. Intersection of three planes in a line

Find the symmetric equation of the line of intersection of the planes

P_1 : x + y - z = 0

P_2 : 2x - y + 4z = -3

P_3 : x + 3y - 5 z = 2

Answer:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & -3 \\ 1 & 3 & -5 & 2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 2 & -4 & 2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From row 2: y = 2z + 1

From row 1: x = z - y = -z - 1

Let z = λ to give the parametric equations x = -1 - λ, y = 1 + 2λ, z = λ

This gives the intersection as a line with symmetric equation $\frac{x+1}{-1} = \frac{y-1}{2} = \frac{z-0}{1}$

3. No intersection of three planes

Show that the planes do not intersect where

P_1 : x + y + z = 2

P_2 : 2x - 2y + z = 5

P_3 : 3x - y + 2z = -1

Answer:

Using Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -2 & 1 & 5 \\ 3 & -1 & 2 & -1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & 1 \\ 0 & -4 & -1 & -7 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & 1 \\ 0 & 0 & 0 & -8 \end{array}\right)$$

This gives 0 = -8, which is impossible. The planes do not intersect at a common point or line.

Intersection of three planes exercise

There is another exercise on the web if you want to try something different.

30 min

Q71: Find the point of intersection of the planes

P_1 = 6x - 9y - 8z = 6

P_2 = 3x - 2y - 9z = -17

P_3 = 6x + 6y - 5z = -15

Q72: P_1 = x + y + z = 6, P_2 = -x + y + 2z = 5 and P_3 = 3x + 2z = 12

Ascertain whether the three planes P_1, P_2 and P_3 as shown intersect. If they do, provide the relevant information on the intersection.

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Q73: If the planes $P_1$with equation x + y + z = 1, P_2 with equation 2x + 2y + z = 3 and P_3 with equation 4x + 4y + 3z = 5 intersect, give details of the intersection.

Q74: Ascertain whether the three planes with equations

4x + y + 5z = 2, -x + y - 2z = 7 and 3x - 3y + 6z = 21 intersect.

If they do provide details of the intersection.

Q75: Find the angle between the planes x + y - 4z = -1 and 2x - 3y + 4z = -5

Q76: Find the acute angle between the planes 2x - y = 0 and x + y + z = 0

1.10 Lines and planes

Learning Objective

Find relationships between lines and planes

The intersection of a line and a plane can be found by substituting the equation of the line in parametric form into the equation of the plane and solving for λ

If there is a unique solution then a point of intersection exists. If there are infinite solutions then the line lies on the plane. If there is no solution then the line is parallel to the plane.

Example : Intersection of a line and a plane

Find the intersection between the plane x - y - 2z = -15 and the line with symmetric equation $\frac{x - 1}{1} = \frac{y - 2}{1} = \frac{z - 3}{2}$

Answer:

In parametric form this line is given by the equations

x = 1 + λ, y = 2 + λ and z = 3 + 2λ

Substitute this into the equation of the plane to give

1 + λ - 2 - λ - 6 - 4λ = -15

i.e. λ = 2

When λ = 2, x = 3, y = 4 and z = 7

Hence the point of intersection is (3, 4, 7)

If a line L and a plane P intersect, it is possible to calculate the angle of intersection between the line and the plane. To do this, find the normal **n** to the plane and find the angle between L and L_2 where L_2 is in the direction **n**. The angle of intersection between the line and the plane is the **complement** of the angle between the line and the normal to the plane.

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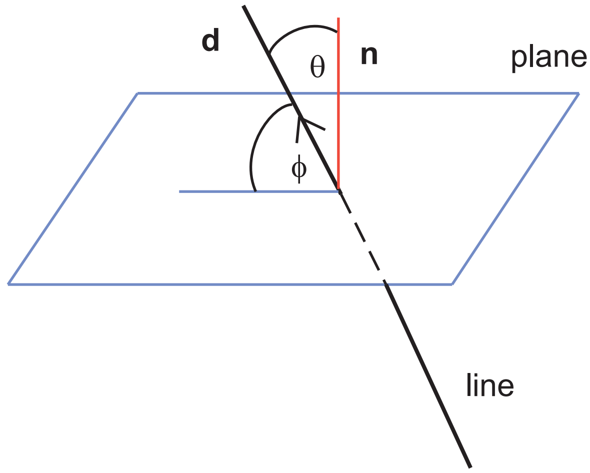


Example : Angle of intersection between a line and a plane

Find the acute angle between the line $\frac{x-1}{4} = \frac{y-2}{3} = \frac{z+3}{2}$

and the plane -10x + 2y - z = 1

Answer:

Take the direction vector of the line, that is, **d** = 4**i** + 3**j** + 2**k**

The normal to the plane is **n** = -10**i** + 2**j** - **k**

Thus $\cos\theta = \frac{\mathbf{n} \bullet \mathbf{d}}{|\mathbf{n}||\mathbf{d}|} = \frac{-36}{\sqrt{105}\sqrt{29}}$ and so $\theta = 130.72°$

Thus the acute angle is 49.28°

The angle between the line and plane is 90 - 49.28 = 40.72°

It is particularly difficult to visualise lines and planes in three dimensions but special isometric graph paper can help.

Activity

Take some isometric graph paper and for a selection of the lines or planes in this topic, try some drawings. Note any intersections and confirm the results given such as parallel planes, skew lines or three planes intersecting at a point.

Interactive lines and planes in space

There is an animation on the web of lines and planes in space to help with the concepts of intersections, angles and parallel properties.

10 min

15 min

Lines, points and planes exercise

There are some questions on the web for you to try if you wish.

Q77: Find the point of intersection between the plane 3x + 2y - z = 5 and the line with symmetric equations $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$

Q78: Find the acute angle between

the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane 10x + 2y - 11z = 3

Q79: Show that the line of intersection of the planes

x + 2y - 2z = 5 and 5x - 2y - z = 0 is parallel to the line $\frac{x+3}{2} = \frac{y}{3} = \frac{z-1}{4}$

Q80: Find a plane through the point (2, 1, -1) and perpendicular to the line of intersection of the planes 2x + y - z = 3 and x + 2y + z = 2

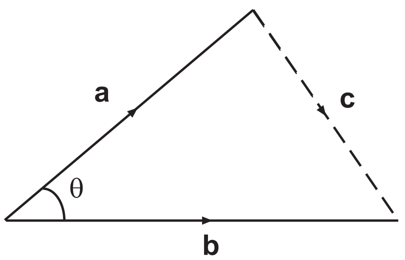
1.11 Summary

At this stage the following techniques and methods should be familiar:

- Using basic vector arithmetic.
- Calculating scalar, vector and scalar triple products.
- Using direction cosines and direction ratios.
- Finding the equation of a line and solving related problems.
- Finding the equation of a plane and solving related problems.
- Finding the intersections of and angles between lines and planes.

1.12 Proofs

Proof 1: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$



Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

c = **b - a**

$$
\begin{aligned}
|\,\mathbf{c}\,|^2 &= |\,\mathbf{b} - \mathbf{a}\,|^2 \\
&= (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 \\
&= b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 + b_3^2 - 2a_3b_3 + a_3^2 \\
\text{Thus} \quad &= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2(a_1b_1 + a_2b_2 + a_3b_3) \\
&= |\,\mathbf{a}\,|^2 + |\,\mathbf{b}\,|^2 - 2(\mathbf{a} \bullet \mathbf{b}) \text{ but} \\
|\,\mathbf{c}\,|^2 &= |\,\mathbf{a}\,|^2 + |\,\mathbf{b}\,|^2 - 2|\,\mathbf{a}\,||\,\mathbf{b}\,| \cos\theta \text{ by the cosine rule so} \\
(\mathbf{a} \bullet \mathbf{b}) &= |\,\mathbf{a}\,||\,\mathbf{b}\,| \cos\theta
\end{aligned}
$$

Proof 2: $|\,\mathbf{a} \times \mathbf{b}\,| = |\,\mathbf{a}\,||\,\mathbf{b}\,| \sin\theta$

$$
\begin{aligned}
|\,\mathbf{a} \times \mathbf{b}\,|^2 &= |\,(\mathbf{a} \times \mathbf{b}) \bullet (\mathbf{a} \times \mathbf{b})\,| \\
&= \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_1b_3 - b_1a_3 \\ a_1b_2 - b_1a_2 \end{pmatrix} \bullet \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_1b_3 - b_1a_3 \\ a_1b_2 - b_1a_2 \end{pmatrix} \\
&= a_2^2b_3^2 - 2a_2b_2a_3b_3 + a_3^2b_2^2 + a_1^2b_3^2 - 2a_1b_1a_3b_3 + a_3^2b_1^2 + a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2 \\
&= (\underline{a_1^2b_1^2} + a_2^2b_1^2 + a_3^2b_1^2 + a_1^2b_2^2 + \underline{a_2^2b_2^2} + a_3^2b_2^2 + a_1^2b_3^2 + a_2^2b_3^2 + \underline{a_3^2b_3^2}) \\
&\quad - (\underline{a_1^2b_1^2} + 2a_1b_1a_2b_2 + \underline{a_2^2b_2^2} + 2a_2b_2a_3b_3 + \underline{a_3^2b_3^2} + 2a_1b_1a_3b_3) \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\,\mathbf{a}\,|^2|\,\mathbf{b}\,|^2 - (\mathbf{a} \bullet \mathbf{b})^2 \\
&= |\,\mathbf{a}\,|^2|\,\mathbf{b}\,|^2 - |\,\mathbf{a}\,|^2|\,\mathbf{b}\,|^2 \cos^2\theta \\
&= |\,\mathbf{a}\,|^2|\,\mathbf{b}\,|^2 (1 - \cos^2\theta) \\
&= |\,\mathbf{a}\,|^2|\,\mathbf{b}\,|^2 \sin^2\theta \\
|\,\mathbf{a} \times \mathbf{b}\,| &= |\,\mathbf{a}\,||\,\mathbf{b}\,| \sin\theta
\end{aligned}
$$

Note the trick of adding in and then subtracting the terms that are underlined.

Proof 3: $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{pmatrix}$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{pmatrix} = a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$

$$\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\mathbf{a} \bullet \mathbf{c} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = a_1c_1 + a_2c_2 + a_3c_3$$

Thus

$$\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} = a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3$$
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$

Proof 4: $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$\mathbf{a} \bullet \mathbf{b}$

$= a_1b_1 + a_2b_2 + a_3b_3$

$= b_1a_1 + b_2a_1 + b_3a_3$ (since ab = ba for any real numbers)

$= \mathbf{b} \bullet \mathbf{a}$

Proof 5: $\mathbf{a} \bullet \mathbf{a} = |\,\mathbf{a}\,|^2 \geqslant 0$

Recall the definition of length.

If $\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ then the length of $\mathbf{p} = \sqrt{a^2 + b^2 + c^2} = |\,\mathbf{p}\,|$

But using the scalar product

$\mathbf{p} \bullet \mathbf{p} = (a \times a) + (b \times b) + (c \times c) = a^2 + b^2 + c^2 = |\,\mathbf{p}\,|^2$

Since any square of a real number is positive then $\mathbf{p} \bullet \mathbf{p}$ is always greater than or equal to zero.

Proof 6: $\mathbf{a} \bullet \mathbf{a} = 0$ if and only if $\mathbf{a} = 0$

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

then $\mathbf{a} \bullet \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = {a_1}^2 + {a_2}^2 + {a_3}^2$

The square of a real number is positive.

So $\mathbf{a} \bullet \mathbf{a} = 0$ only if ${a_1}^2 = {a_2}^2 = {a_3}^2 = 0$

Thus $a_1 = a_2 = a_3 = 0 \Rightarrow \mathbf{a} = 0$

Proof 7: For non-zero vectors, **a** and **b** are perpendicular if and only if $\mathbf{a} \bullet \mathbf{b} = 0$

$\mathbf{a} \bullet \mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta$

If $\mathbf{a} \bullet \mathbf{b} = 0$ then $|\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta = 0$

$\Rightarrow \cos\theta = 0$ since **a** and **b** are non-zero vectors.

Thus $\theta = 90°$ and the vectors **a** and **b** are perpendicular.

Conversely if **a** and **b** are perpendicular then $|\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta = 0$

and so $\mathbf{a} \bullet \mathbf{b} = 0$

Proof 8: $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})$

$\mathbf{b} \times \mathbf{c} = (b_2c_3 - c_2b_3)\mathbf{i} - (b_1c_3 - c_1b_3\mathbf{j} + (b_1c_2 - c_1b_2)\mathbf{k}$

$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = a_1b_2c_3 - a_1c_2b_3 - a_2b_1c_3 + a_2c_1b_3 + a_3b_1c_2 - a_3c_1b_2$

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c x a = $(c_2a_3 - a_2c_3)$**i** - $(c_1a_3 - a_1c_3)$**j** + $(c_1a_2 - a_1c_2)$**k**

b • (c x a) = $b_1c_2a_3 - b_1a_2c_3 - b_2c_1a_3 + b_2a_1c_3 + b_3c_1a_2 - b_3a_1c_3$

a x b = $(a_2b_3 - b_2a_3)$**i** - $(a_1b_3 - b_1a_3)$**j** + $(a_1b_2 - b_1a_2)$**k**

c • (a x b) = $c_1a_2b_3 - c_1b_2a_3 - c_2a_1b_3 + c_2b_1a_3 + c_3a_1b_2 - c_3b_1a_2$

Rearrangement of the terms in each show that they are all equal.

1.13 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web which give a selection of interesting sites to visit. These sites can lead to an advanced study of the topic but there are many areas that will be of passing interest.

Stevin

Simon Stevin was a Flemish mathematician of the 16/17th century. He was an outstanding engineer and used the concept of vector addition on forces. This was however a long time before vectors were generally accepted.

Hamilton

This Irish mathematician was the first, in 1853, to use the term 'vector'.

Gibbs

In the late 1800s, Josiah Gibbs used vectors in his lectures. He made major contributions to the work in vector analysis.

Weatherburn

An Australian, Charles Weatherburn published books in 1921 on vector analysis.

There are many more mathematicians, such as those already mentioned in the complex numbers topic, who played a part in the development of vectors.

1.14 Review exercise

Review exercise in vectors

15 min

Choose two questions to complete in a time of about 15 minutes.

There is a web exercise available for you to try if you prefer it.

Q81: Given the vectors **a** = 3**j** - **k**, **b** = **i** + 2**j** - **k** and **c** = -**i** + 2**j** + 2**k**,

calculate **a x b** and **a • (b x c)**

Q82: Obtain, in parametric form, an equation for the line which passes through the points (2, - 3, 1) and (1, -1, 7)

Q83: Find the equation of the plane which has a normal vector **n** = 2**i** + **j** + 3**k** and passes through the point (1, 2, 3)

Q84: Find the equation of the line of intersection in parametric form of the planes 2x - 3y + z = 3 and 3x + 2y + z = 2

Q85: Find the point of intersection of the planes

P_1 : 3x - 2y - 4z = 3

P_2 : x + y + z = 4

P_3 : -2x - 2y - 3z = -10

1.15 Advanced review exercise

Advanced review exercise in vectors

15 min

There is a similar exercise on the web for you to try if you like.

Q86: Let **u** = **i** - 4**j** - **k** and **v** = 2**i** - 2**j** + **k**.

a) Find **u** •**v** and **u x v**

b) Three planes θ_1, θ_2, θ_3 are given by the equations

θ_1 : x - 4y - z = 3

θ_2 : 2x - 2y + z = 6

θ_3 : 3x - 11y - 2z = 10

i) Find the acute angle between the planes θ_1 and θ_2

ii) By using Gaussian elimination show that the planes θ_1, θ_2 and θ_3 intersect in a point Q and obtain the coordinates of Q

iii) Find an equation for the line L in which θ_1 and θ_2 intersect, and the point R in which L intersects the xy-plane.

c) Three non-zero vectors **a**, **b** and **c** are such that

a x b = c and **b x c = a**

Explain briefly why **a**, **b** and **c** must be mutually perpendicular and why **b** must be a unit vector.

(1996 CSYS paper II)

Q87: P is the point (2, -3, 1) and Q is (-1, 2, -1)

a) Find **p** • **q** and **p x q**

b) Three planes P_1, P_2 and P_3 are given by the equations

P_1 : 4x - 4y + z = 8

P_2 : 2x + 3y - z = -1

P_3 : x + y + z = 0

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i) By Gaussian elimination, or otherwise, find the point of intersection of the three planes.

ii) Find the equation of the line of intersection of the planes P_2 and P_3

iii) State where it meets the yz-plane.

Q88: a) Find the equation of the plane through the points

P (1, -2, 6), Q (1, 0, 3) and R (2, 1, 2) in Cartesian form.

b) Find the angle QPR

c) Find the angle between the plane and the line with parametric equations

$x = 2\lambda - 3$, $y = 3\lambda$, $z = 4\lambda + 1$

1.16 Set review exercise

Set review exercise

15 min

The answers for this exercise are only available on the web by entering the answers obtained in an exercise called 'set review exercise'. The questions may be structured differently but will require the same answers.

Q89: Let **p** = 3**i** - **j** - **k** and **q** = -**i** + 3**k**Find **q x p** and **p • q**

Q90: Find the vector, parametric and symmetric forms of the equation of the line which passes through the points (-2, 1, -1) and (3, 2, -2)

Q91: Give the equation of the plane with normal vector **n** = **i** + 2**j** - 2**k** and which passes through the point (1, -3, 1)

Topic 2

Matrix algebra

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Learning Objectives

- *Use matrix algebra.*

Minimum Performance Criteria:

- *Perform the matrix operations of addition, subtraction and multiplication.*
- *Calculate the determinant of a 3 $\times$ 3 matrix.*
- *Find the inverse of a 2 $\times$ 2 matrix.*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Algebraic manipulation.
- Basic matrix concepts.
- Transformations of functions.

2.1 Revision exercise

Learning Objective

Identify areas that need revision

Revision exercise

20 min

If any question in this revision exercise causes difficulty then it may be necessary to revise the techniques before starting this topic. Ask a tutor for advice.

Q1: Name the element a_{23} in the matrix $\begin{pmatrix} 2 & 7 & 4 \\ -3 & -2 & 1 \end{pmatrix}$

Q2: Using elementary row operations, solve the system of equations

$2x - y + 2z = 11$

$x + y + z = 4$

$3x - y - 2z = 1$

Q3: Give the coordinates of the point (2, 3) under the following transformations:

1. under reflection in the x-axis.
2. under rotation by 90° anticlockwise.

Q4: Find the vector product of the two vectors **a** and **b**

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} - a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} - b_2\mathbf{j} - b_3\mathbf{k}$

2.2 Basic matrix terminology

Learning Objective

Know the basic terminology used in matrix algebra

The following terms are worth emphasing at this point as they will be used frequently in this topic.

Matrix

A matrix is a rectangular array of numbers.

Example

$\begin{pmatrix} 2 & -3 \\ 1 & 5 \\ 3 & -1 \end{pmatrix}$ This matrix has 3 **rows**, 2 **columns** and 6 **elements**.

The order of this matrix is taken from the number of rows and the number of columns.

In this case the matrix has **order** 3 x 2

In general terms, an m $\times$ n matrix will have m rows, n columns and mn elements.

It can be shown as $A = \begin{pmatrix} a_{11} & a_{12} & \cdot\cdot & a_{1n} \\ a_{21} & a_{22} & \cdot\cdot & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{m1} & a_{m2} & \cdot\cdot & a_{mn} \end{pmatrix}$ $\updownarrow$ m rows

$\leftrightarrow$

n columns

This matrix A is written as A = $(a_{ij})_{m\times n}$

a_{ij} is the element in the i-th row and the j-th column.

Here are examples of matrices:

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 & -2 \\ 0 & 5 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Column matrix

A column matrix has only one column but any number of rows.

The matrix $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is called a **column** matrix.

Row matrix

A row matrix has only one row but any number of columns.

The matrix $\begin{pmatrix} 1 & 2 \end{pmatrix}$ is called a **row** matrix.

Square matrix

A square matrix has the same number of rows and columns.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a **square matrix**. This particular example has a special property which will be explained in the section on determinants and inverses.

2.3 Matrix operations

Learning Objective

Perform addition, subtraction and multiplication operations on matrices

It is possible to carry out basic arithmetic operations such as addition and subtraction with matrices.

In fact, matrix addition has already been demonstrated in a previous topic. Recall the following example from topic 11 on vectors.

'If **p** = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and **q** = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are vectors then **p + q** is the vector $\begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}$'

The ordered triple form of the vector is, of course, a 3 x 1 matrix and this example shows the matrix addition of two 3 x 1 matrices.

Note: only matrices of the same order can be added (or subtracted). The technique is to add the corresponding ijth elements together.

Matrices which have the same order are said to be conformable for addition (or subtraction).

Expanding on this, matrix addition can be defined as follows:

Addition of two matrices

If A and B are two matrices of the same order, the matrix A + B is formed by adding the corresponding elements of each matrix.

Example : Addition of matrices

Add the following matrices

1. $\begin{pmatrix} -2 & 3 \\ -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$
2. $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$
3. $\begin{pmatrix} -1 & 0 & -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & -2 & 1 \end{pmatrix}$

Answer:

1. $\begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix}$
2. $\begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & -3 & 2 \end{pmatrix}$

Matrix adding examples

There are also examples of matrix addition on the web.

10 min

Adding matrices exercise

There is another exercise on the web for you to try if you wish.

5 min

Q5: Add $\begin{pmatrix} -2 & -4 \\ 2 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$

Q6: Add $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

Q7: Add $\begin{pmatrix} -1 & 9 & -4 \\ 2 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} -3 & 2 & 1 \\ 0 & -4 & 3 \\ -1 & 1 & -2 \end{pmatrix}$

Q8: Add $\begin{pmatrix} -2 & 3 \\ -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 3 & -3 \\ 1 & 2 \end{pmatrix}$

Subtraction is just as straightforward as addition. Note that again the matrices must have the same order.

Subtraction of two matrices

If A and B are two matrices of the same order, the matrix A - B is formed by subtracting the corresponding elements of matrix B from those in matrix A.

Example : Subtraction of matrices

Subtract the following matrices:

1. $\begin{pmatrix} -2 & 3 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$
2. $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$
3. $\begin{pmatrix} -1 & 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & -2 & 1 \end{pmatrix}$

Answer:

1. $\begin{pmatrix} -3 & 4 \\ -3 & 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$
3. $\begin{pmatrix} -2 & -1 & 1 & 0 \end{pmatrix}$

10 min

Matrix subtracting examples

There are also examples of matrix subtraction on the web.

5 min

Subtracting matrices exercise

There is another exercise on the web for you to try if you wish.

Q9: $\begin{pmatrix} -3 & -4 \\ -3 & -1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$

Q10: $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

Q11: $\begin{pmatrix} -1 & 4 & -2 \\ 2 & -1 & -1 \\ 0 & 3 & 2 \end{pmatrix} - \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 3 \\ -1 & 3 & -2 \end{pmatrix}$

Q12: $\begin{pmatrix} -2 & 3 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & -3 \\ 1 & 2 \end{pmatrix}$

Matrices can be multiplied by a scalar. Again, in topic 11 on vectors this was demonstrated. Recall the paragraph:

'Vectors can also be multiplied by a scalar.

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and λ is a real number, (a scalar) then $\lambda\mathbf{p} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$'

This technique is used for a matrix of any order.

Multiplication by a scalar

To multiply an m x n matrix A by a scalar λ, take each element and multiply it by λ to give the matrix λA.

Example : Multiplication of a matrix by a scalar

Multiply the matrix $\begin{pmatrix} -3 & 0 & 1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{pmatrix}$ by -3

Answer:

$$-3 \times \begin{pmatrix} -3 & 0 & 1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & -3 \\ -6 & -3 & 6 \\ 9 & -3 & -3 \end{pmatrix}$$

From the previous example it is clear to see that for any matrix A multiplied by a scalar λ then $\lambda A = A\lambda$

Note also that the new matrix λA has the same order as the original matrix A.

Multiplication of a matrix by a scalar exercise

There is a similar exercise on the web for you to try if you prefer.

5 min

Q13: Find the matrix formed by λA where λ is -2 and A = $\begin{pmatrix} -2 & -1 & 3 \end{pmatrix}$

Q14: State the value of λ when

A = $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 1 & -2 \end{pmatrix}$ and Aλ = $\begin{pmatrix} -2 & 1 & -3 \\ 1 & -1 & 2 \end{pmatrix}$

Q15: If A = $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 3 & -2 & 4 \end{pmatrix}$ find the matrix λA which has element $_{23}$ = 8

The final matrix operation to consider in this section is matrix multiplication.

Two matrices can only be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second matrix.

When this is possible the two matrices are said to be **conformable for multiplication**.

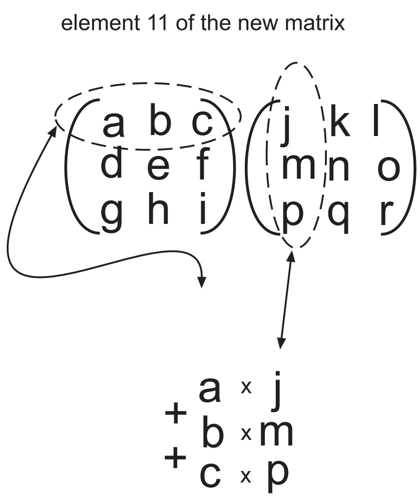
Example : Matrices conformable for multiplication

$\begin{pmatrix} .. & .. & .. \end{pmatrix}$ times $\begin{pmatrix} .. \\ .. \\ .. \end{pmatrix}$ is possible since the first matrix (of order 3 x 1) has 3 column and the second matrix (of order 1 x 3) has 3 rows.

However $\begin{pmatrix} .. & .. & .. \\ .. & .. & .. \end{pmatrix}$ times $\begin{pmatrix} .. & .. & .. \\ .. & .. & .. \end{pmatrix}$ is impossible since the first matrix has 3 columns but the second matrix only has 2 rows.

It is worth remembering to multiply rows into columns.

The first element of the product of two 3 x 3 matrices is shown on the diagram. This structure may help to make it easier to remember.



The formal definition of matrix multiplication follows.

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Multiplication of matrices

The product of the two matrices A_{mn} and $B_{np} = C_{mp}$

If $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$

then the matrix AB =

$$\begin{pmatrix} a_{11}b_{11} + \dots + a_{1n}b_{n1} & a_{11}b_{12} + \dots + a_{1n}b_{n2} & \dots & a_{11}b_{1p} + \dots + a_{1n}b_{np} \\ a_{21}b_{11} + \dots + a_{2n}b_{n1} & a_{21}b_{12} + \dots + a_{2n}b_{n2} & \dots & a_{21}b_{1p} + \dots + a_{2n}b_{np} \\ \dots & \dots & \dots & \dots \\ a_{m1}b_{11} + \dots + a_{mn}b_{n1} & a_{m1}b_{12} + \dots + a_{mn}b_{n2} & \dots & a_{m1}b_{1p} + \dots + a_{mn}b_{np} \end{pmatrix}$$

Some examples will help to make this clearer.

Example : Matrix multiplication

Find AB where $A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & -2 \\ 3 & 1 & 0 \end{pmatrix}$

Answer:

Since matrix A has order 3 x 2 and matrix B has order 2 x 3 the resulting matrix AB will have order 3 x 3

$$AB = \begin{pmatrix} (-1 \times -1 + 2 \times 3) & (-1 \times 2 + 2 \times 1) & (-1 \times -2 + 2 \times 0) \\ (0 \times -1 + 3 \times 3) & (0 \times 2 + 3 \times 1) & (0 \times -2 + 3 \times 0) \\ (-2 \times -1 + 1 \times 3) & (-2 \times 2 + 1 \times 1) & (-2 \times -2 + 1 \times 0) \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 2 \\ 9 & 3 & 0 \\ 5 & -3 & 4 \end{pmatrix}$$

Multiplying matrices exercise

10 min

There is a web exercise for you to try if you prefer it.

Q16: Multiply the matrices $\begin{pmatrix} -1 & 2 & -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$

Q17: Multiply $\begin{pmatrix} -1 & 0 & -2 \\ 3 & 2 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 & -1 & 1 \\ 4 & -3 & 2 & -2 \\ 2 & 0 & 1 & 1 \end{pmatrix}$

Q18: For the four matrices A, B, C, D, form all the possible products.

$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 0 & 3 \end{pmatrix}$,

$C = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ -3 & 2 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 1 & 1 & -2 \\ 1 & 1 & 0 & -1 \\ 2 & -2 & 1 & 2 \end{pmatrix}$

Q19: Multiply $\begin{pmatrix} -2 & 1 & 0 & 2 \\ 3 & -2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}$ by $\begin{pmatrix} 0 & 1 & 2 & 0 \\ -2 & 1 & -1 & 3 \\ 2 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Note carefully:

Two matrices are equal if and only if the following two conditions apply:

- They have the same order.
- Each corresponding element ij is equal.

Example : Equality of matrices

Find the values of x and y which make the two matrices A and B equal when

A = $\begin{pmatrix} x+2 & 4 \\ -1-y & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & x+3 \\ 1 & 2x+y \end{pmatrix}$

Answer:

Both matrices are 2 x 2

Equate each element in turn

$x + 2 = 3$

$4 = x + 3$

$-1 - y = 1$

$0 = 2x + y$

and solve these to give x = 1 and y = -2

Thus if x = 1 and y = -2 both matrices represent the matrix $\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$

If there are no values of x and y which satisfy all the equations then the matrices are not equal.

Q20: Are the following pairs of matrices equal:

1. $\begin{pmatrix} x & 4 \\ y-2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2-x & x+3 \\ 4 & y \end{pmatrix}$
2. $\begin{pmatrix} x & 4 \\ y-2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2-x & x+3 \\ 4 & y-4 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{pmatrix}$
4. $\begin{pmatrix} x-2 & 4 \\ x & 2 \\ 1 & y-1 \end{pmatrix}$ and $\begin{pmatrix} 1 & x+1 \\ 3 & y+2 \\ y+1 & -1 \end{pmatrix}$

2.4 Determinants and inverses

Learning Objective

Find and use the determinant and the inverse of a matrix

2.4.1 Determinants

The word 'determinant' may be familiar. If you have studied vectors consider the vector product . The work which now follows should clarify the relationship between the vector product and the determinant.

Determinant of a matrix

The determinant of a matrix is a value representing sums and products of a square matrix.

If the matrix A = $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

then the determinant of A is det A = $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

Note that the determinant only exists for a square matrix and that the determinant is a number, not a matrix.

Determinant value of a 2 x 2 matrix

If A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the value of det A = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad - bc

Example Find the determinant of the matrix A = $\begin{pmatrix} -1 & -2 \\ 2 & -3 \end{pmatrix}$

Answer:

det A = ad - bc = (-1 $\times$ -3) - (-2 $\times$ 2) = 3 + 4 = 7

5 min

2 x 2 determinant exercise

There is another exercise on the web for you to try if you prefer.

Q21: Find the determinant of the following matrices:

a) $\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$

c) $\begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

e) $\begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$

The calculation of the determinant of a 3 x 3 matrix is slightly more complicated and involves the technique previously shown for the 2 x 2 matrices.

There are two methods. The one explained here is called the Laplace expansion.

Determinant value of a 3 x 3 matrix

If A = $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then the value of det A =

$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} =$

a(ei - hf) - b(di - gf) + c(dh - ge)

Remember the minus sign for the middle term in this calculation.

Compare this formula with that for the vector product :

'If **p** = a**i** + b**j** + c**k** and **q** = d**i** + e**j** + f**k**

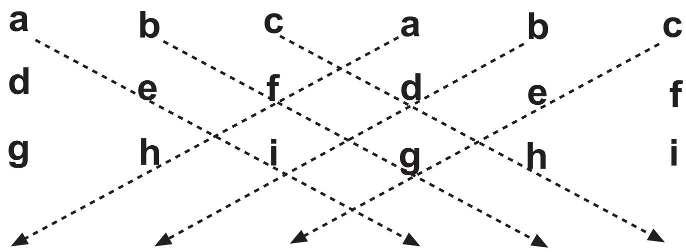
then **p x q** = (bf - ec)**i** - (af - dc)**j** + (ae - db)**k**

This is actually the **determinant** of the matrix $\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$'

Important: The same formula is used but the vector product is a vector whereas the determinant of a matrix is a value.

There is an easy way to remember the calculation for the determinant of a 3 x 3 matrix.

- Make two copies of the matrix side by side.
- For each entry in the top row of the first matrix multiply down the diagonal to the right.
- For each entry in the top row of the second copy of the matrix, multiply down the diagonal to the left and add a minus sign.



det = -afh -bdi -ceg +aei +bfg +cdh

Example Find the determinant of $\begin{pmatrix} 2 & -1 & 2 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

Answer:

The determinant is

$2\ (0 \times 2 - 1 \times 1) - (-1)(-3 \times 2 - 0 \times 1) + 2(-3 \times 1 - 0 \times 0) = -2 - 6 - 6 = -14$

10 min

3 x 3 determinant exercise

There is a web version of this exercise for you to try if you prefer.

Q22: Find the determinant of the following matrices:

a) $\begin{pmatrix} -2 & 1 & 3 \\ 0 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 3 & -2 & 0 \\ 3 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} -2 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$

Consider the matrix $\begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix}$

This has a determinant of 0 and matrices such as this have a special name.

Singular matrix

A matrix whose determinant is zero is called a singular matrix.

Conversely, any matrix which has a non-zero determinant is said to be non-singular.

2.4.2 Inverses

Although the formula for a determinant can be used to find the vector product, its main use is in finding the inverse of a matrix.

In basic arithmetic for any number n there is a multiplicative inverse ($^1/_n$) such that $n \times {}^1/_n = 1$

A multiplicative inverse also exists for a matrix if it satisfies two conditions.

1. The matrix must be square.
2. The determinant of the matrix must be non-zero.

When these conditions are satisfied the matrix is said be to **invertible**.

Inverse of a 2 x 2 matrix

Let A be the square non-singular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The inverse of A is denoted $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The definition clearly shows why the matrix has to be non-singular.

If the determinant is zero then $\frac{1}{ad - bc} = \frac{1}{0}$ which is undefined.

When the inverse is calculated it is customary either to leave the fraction outside or, if taken inside, to leave each element of the matrix as a fraction.

Example : Inverse of a 2 x 2 matrix

Find the inverse of the matrix $A = \begin{pmatrix} -2 & 2 \\ 3 & 4 \end{pmatrix}$

Answer:

The inverse exists since the matrix A is a square non-singular matrix.

det A = $(-2 \times 4) - (3 \times 2) = -14$

$a^{-1} = \frac{1}{-14}\begin{pmatrix} 4 & -2 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-2}{7} & \frac{1}{7} \\ \frac{3}{14} & \frac{1}{7} \end{pmatrix}$

Inverse example

There is an animation of the construction of an inverse on the web for you to look at if you wish.

5 min

2 x 2 inverse exercise

There is a web exercise for you to try if you prefer.

10 min

Q23: Find the inverses of the following matrices:

a) $\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$

c) $\begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 2 & 2 \\ -3 & 3 \end{pmatrix}$

In the same way that it is possible to check that an integer times its inverse is equal to one (e.g. $3 \times {}^{1}/_{3} = 1$), it is possible to check that the matrix times its inverse gives the matrix equivalent of 1. The matrix equivalent of 1 is called the identity matrix.

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Identity matrix

An n x n matrix which has only ones on the leading diagonal and zeros elsewhere is called an identity matrix and takes the form

$$\underbrace{\begin{pmatrix} 1 & 0 & .. & 0 \\ 0 & 1 & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & 1 \end{pmatrix}}_{\leftarrow n \rightarrow} \updownarrow n$$

It is denoted by I.

The structure of this matrix may look rather familiar.

In vectors the standard basis vectors can be shown as the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

This in fact is an identity matrix. It is the 3 x 3 identity matrix.

Q24: What is the 2 x 2 identity matrix?

Example A previous example showed the inverse of A where

$A = \begin{pmatrix} -2 & 2 \\ 3 & 4 \end{pmatrix}$ as $A^{-1} = \begin{pmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{14} & \frac{1}{7} \end{pmatrix}$

Checking AA^{-1} gives $\begin{pmatrix} -2 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{14} & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Activity

Calculate AA^{-1} and verify that this gives the identity matrix shown. Find $A^{-1}A$ and make a conclusion.

In all cases this is a sensible simple check to make when an inverse of any matrix has been calculated.

The easiest method to use for finding the inverse of a 3 x 3 matrix relies on the elementary row operations that were covered in topic 5 - Systems of linear equations.

Strategy for finding the inverse of a 3 x 3 matrix

- Form an augmented matrix $(A \mid I)$ ($\left(\begin{array}{c|ccc} & 1 & 0 & 0 \\ A & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right)$) using the given 3 x 3 matrix (A) at the left and the 3 x 3 identity matrix (I) at the right.
- Perform elementary row operations to reduce the left-hand matrix to the 3 x 3 identity matrix (Gauss-Jordan elimination).
- The effect on the right matrix will be to convert it into the inverse required.

As a brief reminder, here is the definition of elementary row operations.

Elementary row operations

The three ways in which a matrix can be manipulated to solve a system of equations are called elementary row operations. They are:

- Interchange two rows.
- Multiply one row by a non-zero constant.
- Change one row by adding a multiple of another row.

Elementary row operations

10 min

There are also examples of the application of each operation on the web.

It is important to carry out the elementary row operations in such a way that the identity matrix is formed on the left hand side.

3 x 3 inverse exercise

15 min

There is another version of this exercise on the web for you to try if you like.

Q25: By using ERO's find the inverses for the following matrices:

1. $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 8 & -3 \\ 2 & -5 & 2 \end{pmatrix}$
2. $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 3 & -2 & 1 \end{pmatrix}$

In topic 5, elementary row operations were used to solve a system of three equations in three unknowns say, x, y and z.

The coefficients of these unknowns were used to form a matrix of the coefficients, say A.

The system can then be represented by the equation AX = B where X is the column matrix of the unknowns x, y, and z and B is the column matrix of the constants.

If the augmented matrix $(A \mid I)$ is subsequently converted by elementary row operations to $\left(I \mid A^{-1} \right)$ this can be used to find a solution to this system of equations. A little bit of algebra completes the picture.

AX = B (the system of equations)

Multiply both sides by A^{-1} to give $A^{-1}AX = A^{-1}B$

But $A^{-1}A = I$ and so $X = A^{-1}B$

Thus multiplying the column matrix of constants by the inverse of the matrix of coefficients gives the column matrix of values for the unknowns.

An example will demonstrate this technique.

Example Find the solution to the system of linear equations using the matrix method, where the equations are:

x + 2y + z = 1

2x + 5y + 2z = 3

2x + 4y + z = 5

Answer:

Find the matrix of coefficients, say A.

This gives A = $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 4 & 1 \end{pmatrix}$

By using EROs on the augmented matrix $\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right)$

the inverse A^{-1} = $\begin{pmatrix} 3 & -2 & 1 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

Let X be the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then AX = B where B = $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

This equation (AX = B) represents the system of equations.

so $A^{-1}AX = A^{-1}B$

But $A^{-1}A = I$

so $X = A^{-1}B$

Thus $X = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

The values are $x = 2$, $y = 1$ and $z = -3$

This method is an alternative to the Gaussian elimination method but it is doubtful whether it is any quicker or easier to use.

The determinant however can indicate if a system of equations has a unique solution or not.

If an inverse exists, that is, if the matrix of coefficients is square and non-singular then there is a unique solution to the system.

This determinant check can also be used to determine the nature of an intersection of three planes in three-dimensional space.

If the determinant exists, then the intersection of the planes is at a point. It can also indicate, by the same argument, that three lines do intersect in three-dimensional space.

It is important not to confuse the inverse matrix A^{-1} explained earlier with the transpose of a matrix.

Transpose of a matrix

The transpose of a matrix A of order $m \times n$ is found by reflecting the matrix in its main diagonal.

It is denoted A^T or A'.

The effect is to interchange the rows of the matrix with the columns and A^T has order $n \times m$.

In contrast to an inverse, a transpose exists for any matrix; the matrix does not have to be square or non-singular.

Example : Transpose of a matrix

Find A^T when $A = \begin{pmatrix} 2 & 0 \\ -1 & -4 \\ 1 & 3 \end{pmatrix}$

Answer:

$A^T = \begin{pmatrix} 2 & -1 & 1 \\ 0 & -4 & 3 \end{pmatrix}$

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10 min

Tranposing matrices

There are four examples of forming the transpose on the web.

For some square matrices $A^T = A^{-1}$. These matrices have a special name.

Orthogonal matrix

A square matrix A is orthogonal $\Leftrightarrow A^T = A^{-1}$

$A^T = A^{-1} \Rightarrow \det A = \pm 1$

This information will be useful in the section on transformations.

2.5 Properties of matrices

Learning Objective

Use the properties of matrix operations

There are several interesting results which can be found from examining the operations on matrices. These are rather similar in some cases to the properties of operations on real numbers. Some of these may seem obvious.

The following properties are assumed:

- $AI = A$
- $AA^{-1} = I$

In each of the properties mentioned A, B and C are matrices with general elements of a_{ij} b_{ij} and c_{ij} respectively.

Property 1: $A + B = B + A$

Condition: A and B must have the same order for addition.

For any two real numbers a_1 and b_1 the sum $a_1 + b_1 = b_1 + a_1$

The matrix $(A + B)$ will have elements $(a + b)_{ij}$ and $(B + A)$ has elements $(b + a)_{ij}$

But $(a + b) = (b + a)$ and so each element of $A + B$ and $B + A$ are equal.

Thus $A + B = B + A$

Example Show that $A + B = B + A$

when $A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \\ -4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ 1 & 0 \\ 3 & 2 \end{pmatrix}$

Answer:

$$A + B = \begin{pmatrix} -2 + (-3) & 1 + 2 \\ 0 + 1 & 3 + 0 \\ -4 + 3 & -2 + 2 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 1 & 3 \\ -1 & 0 \end{pmatrix}$$

and $B + A = \begin{pmatrix} -3+(-2) & 2+1 \\ 1+0 & 0+3 \\ 3-4 & 2-2 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ 1 & 3 \\ -1 & 0 \end{pmatrix}$ as required.

Property 2: $AB \neq BA$ in general - this is the commutative property.

Condition: to form either AB or BA the matrices A and B must be square, otherwise both products cannot be formed.

Suppose that A and B have order $m \times m$ and that A and B are not identical. (That is, each element of A $\neq$ each element in B.)

Consider a general element e_{rs} in each product.

For AB the element $e_{rs} = (a_{r1}b_{1s} + a_{r2}b_{2s} + \ldots a_{rm}b_{ms})$

However for BA the element $e_{rs} = (b_{r1}a_{1s} + b_{r2}a_{2s} + \ldots b_{rm}a_{ms})$

Since A and B are not identical, it follows that there is at least one element e_{pq} such that $a_{pm}b_{mq} \neq b_{pm}a_{mq}$

Thus $AB \neq BA$ in general.

Example Show that if $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$ then $AB \neq BA$

Answer:

$AB = \begin{pmatrix} -6 & -1 \\ -4 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} -4 & -7 \\ -2 & -1 \end{pmatrix}$

Property 3: $(AB)C = A(BC)$ - this is the associative property.

Condition: for either product to exist, the orders of the matrices must be

$A = m \times n$, $B = n \times p$, $C = p \times q$

This property follows from the fact that for any three real numbers a, b and c, $(ab)c = a(bc)$ and so each element of ABC can be expressed in either form. The general form of this proof is cumbersome and shown, in the section headed Proofs near the end of this topic, as Proof 1.

Example If $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 3 \\ -2 & -1 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$ show that $(AB)C = A(BC)$

Answer:

$AB = \begin{pmatrix} 5 & 2 & 1 & 5 \\ -2 & 0 & -1 & 5 \end{pmatrix}$ and

$(AB)C = \begin{pmatrix} 5 & 2 & 1 & 5 \\ -2 & 0 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \end{pmatrix}$

BC = $\begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix}$ and A(BC) = $\begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \end{pmatrix}$ as required.

Property 4: A(B + C) = AB + AC - this is the distributive property

Condition: B and C must have the same order and the number of columns in A must equal the number of rows in B and C

The proof is shown for interest in the section headed Proofs near the end of this topic as Proof 2.

Example Let A = $\begin{pmatrix} 1 & -2 \\ -1 & 3 \\ 0 & 2 \end{pmatrix}$, B = $\begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$ and C = $\begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}$ and show that A(B + C) = AB + AC

Answer:

B + C = $\begin{pmatrix} -3 & 3 \\ 1 & -2 \end{pmatrix}$ and

A(B + C) = $\begin{pmatrix} 1 & -2 \\ -1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -3 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -5 & 7 \\ 6 & -9 \\ 2 & -4 \end{pmatrix}$

AB = $\begin{pmatrix} -7 & 3 \\ 9 & -4 \\ 4 & -2 \end{pmatrix}$ and AC = $\begin{pmatrix} 2 & 4 \\ -3 & -5 \\ -2 & -2 \end{pmatrix}$

so AB + AC = $\begin{pmatrix} -7 & 3 \\ 9 & -4 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ -3 & -5 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} -5 & 7 \\ 6 & -9 \\ 2 & -4 \end{pmatrix}$ as required.

Property 5: $(A')' = A$ or $(A^T)^T = A$

Let A be the m $\times$ n matrix $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$

Thus $A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix}$. Call this the matrix B

But $B^T = (A^T)^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ = A as required.

$(A^T)^T = A$

Example Verify that $(A^T)^T = A$ when $A = \begin{pmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 0 \end{pmatrix}$

Answer:

$A^T = \begin{pmatrix} 1 & 3 & 2 \\ -2 & -1 & 0 \end{pmatrix}$ and transposing this matrix gives $\begin{pmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 0 \end{pmatrix}$

So $(A^T)^T = A$

Property 6: $(A + B)^T = A^T + B^T$

Condition: A and B have the same order for addition to take place.

Suppose that A and B are $m \times n$ matrices.

Then $(A + B)^T =$

$$\begin{pmatrix} a_{11} + b_{11} & a_{21} + b_{21} & \dots & a_{n1} + b_{n1} \\ a_{12} + b_{12} & a_{22} + b_{22} & \dots & a_{n2} + b_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} + b_{1m} & a_{2m} + b_{2m} & \dots & a_{nm} + b_{nm} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{21} & \dots & b_{n1} \\ b_{12} & b_{22} & \dots & b_{n2} \\ \dots & \dots & \dots & \dots \\ b_{1m} & b_{2m} & \dots & b_{nm} \end{pmatrix}$$

$= A^T + B^T$

Example Show that $(A + B)^T = A^T + B^T$

when $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \\ -4 & -3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 3 \\ -1 & -2 & -1 \\ -1 & 2 & 0 \end{pmatrix}$

Answer:

$A + B = \begin{pmatrix} 3 & 1 & 5 \\ -2 & 1 & -3 \\ -5 & -1 & 0 \end{pmatrix}$ and

$(A + B)^T = \begin{pmatrix} 3 & -2 & -5 \\ 1 & 1 & -1 \\ 5 & -3 & 0 \end{pmatrix}$

But $A^T = \begin{pmatrix} 1 & -1 & -4 \\ 0 & 3 & -3 \\ 2 & -2 & 0 \end{pmatrix}$ and $B^T = \begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 2 \\ 3 & -1 & 0 \end{pmatrix}$

so $A^T + B^T = \begin{pmatrix} 1 & -1 & -4 \\ 0 & 3 & -3 \\ 2 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 2 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -5 \\ 1 & 1 & -1 \\ 5 & -3 & 0 \end{pmatrix}$

as required.

Property 7: $(AB)^T = B^T A^T$

Condition: A and B must be conformable for multiplication for the product AB to exist.

This is another cumbersome proof, which has been placed in the Proof section near the end of this topic as Proof 3.

Example If $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$ show that $(AB)^T = B^T A^T$

Answer:

$AB = \begin{pmatrix} -4 & -7 \\ -2 & -5 \end{pmatrix}$ so $AB^T = \begin{pmatrix} -4 & -2 \\ -7 & -5 \end{pmatrix}$

But $B^T = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ and $A^T = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$

which gives $B^T A^T = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -7 & -5 \end{pmatrix}$ as required.

Property 8: $(AB)^{-1} = B^{-1} A^{-1}$

Condition: A and B must be square non-singular matrices and conformable for mutliplication. (They are therefore of the same order.)

$(AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= A\, IA^{-1}$ since $BB^{-1} = I$

$= AA^{-1}$ since $AI = A$

$= I$

Thus $B^{-1}A^{-1}$ is the inverse of AB, but the inverse of AB is $(AB)^{-1}$

Therefore $(AB)^{-1} = B^{-1}A^{-1}$

Example For $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ show that $(AB)^{-1} = B^{-1}A^{-1}$

Answer:

$AB = \begin{pmatrix} 4 & -2 \\ 5 & -1 \end{pmatrix}$ and det A = 6

Thus $(AB)^{-1} = \frac{1}{6} \begin{pmatrix} -1 & 2 \\ -5 & 4 \end{pmatrix}$

det A = 2 and det B = 3 so $A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ and $B^{-1} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Thus $B^{-1}A^{-1} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$ times $\frac{1}{2} \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -1 & 2 \\ -5 & 4 \end{pmatrix}$ as required.

Property 9: det (AB) = det A det B

Condition: A and B must be conformable for multiplication.

The proof is shown for 2 x 2 matrices but can be extended to 3 x 3 and larger matrices.

Let A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and B = $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

AB = $\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$

det A = ad - bc, det B = eh - fg and

det A det B = (ad - bc)(eh - fg) = adeh - bceh + bcfg - adfg

det (AB) = (ae + bg)(cf + dh) - (ce + dg)(af + bh)

= acef + bcfg + adeh + bdgh - acef - adfg - bceh - bdgh

= bcfg + adeh - bceh - adfg

Thus det A det B = det (AB)

Activity

Verify the property for 3 x 3 matrices.

Properties exercise

There is an alternative exercise on the web for you to try if you wish.

15 min

Q26: Show that A + B = B + A

when A = $\begin{pmatrix} -2 & 3 \\ 0 & -2 \\ 1 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} -2 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$

Q27: Show that if A = $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$ then AB $\neq$ BA

Q28: If A = $\begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix}$, B = $\begin{pmatrix} -1 & 1 & 0 & -1 \\ -2 & 0 & 1 & 3 \\ -2 & 1 & 0 & 2 \end{pmatrix}$ and C = $\begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}$ show that (AB)C = A(BC)

Q29: Let A = $\begin{pmatrix} 2 & -1 \\ -3 & 0 \\ 1 & 1 \end{pmatrix}$, B = $\begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$ and C = $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$ and show that A(B + C) = AB + AC

Q30: Verify that $(A^T)^T$ = A when A = $\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ 4 & 2 \end{pmatrix}$

Q31: Show that $(A + B)^T = A^T + B^T$

when $A = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 1 & 1 \\ -3 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & -2 \\ 2 & 0 & 1 \\ -2 & 3 & 1 \end{pmatrix}$

Q32: If $A = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ show that $(AB)^T = B^T A^T$

Q33: For $A = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 \\ -2 & -1 \end{pmatrix}$ show that $(AB)^{-1} = B^{-1}A^{-1}$

2.6 Matrices and transformations

Learning Objective

Identify the relationship between matrix operations and transformations in the (x, y) plane

Transformations in the plane, other than translations, can be represented by 2 x 2 matrices.

Any point (x, y) can be mapped to the point (x^1, y^1) with a 2 x 2 matrix such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^1 \\ y^1 \end{pmatrix}$ In fact a **linear transformation** maps any line in the x-y plane to a line in the plane such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^1 \\ y^1 \end{pmatrix}$ where $ad - bc \neq 0$

Transformations include:

- rotations
- reflections
- dilatations (commonly known as scalings).

Rotations

Consider the effect of taking the point (5, 2) and rotating it by 90° in an anticlockwise direction about the origin.

After rotation the image of the point is at coordinates Q = (-2, 5)

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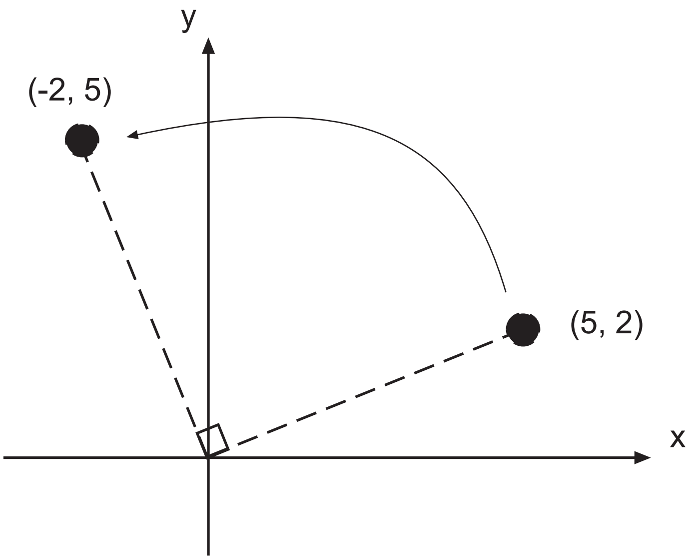


The point P (5, 2) can be represented by the 2 x 1 (column) matrix A = $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

If this matrix is then multiplied by the 2 x 2 matrix R = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the result is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

In matrix terms RA = B where B is the matrix which represents the point Q, the coordinates of the image of P under the rotation.

Q34: Find the simplest matrix which will rotate this point (5, 2) through -90°

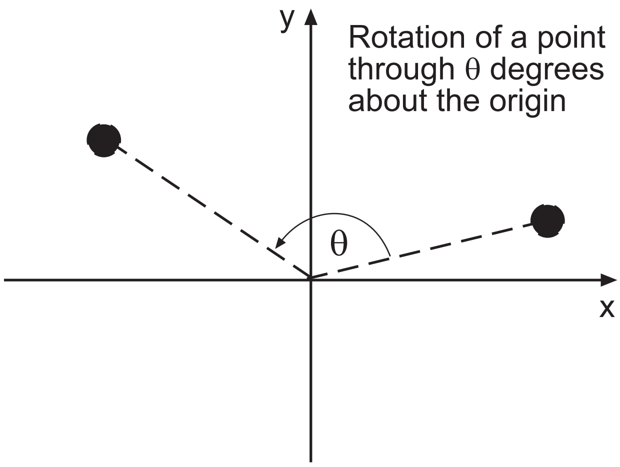
The angle of rotation determines the transformation matrix through values of the cosine and sine of this angle.

Activity

Before continuing further, examine the cos and sin of the angles 90° and -90° in relationship to the two rotation matrices shown earlier. Try to deduce the form of a general matrix of rotation.

Rotation matrix

The matrix R = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ acting, by multiplication on the column matrix A = $\begin{pmatrix} x \\ y \end{pmatrix}$, represents the effect of rotation through $\theta°$ anticlockwise about the origin on the point (x, y)



Example : Rotation of a point

Find the coordinates of the point (3, 4) under an anticlockwise rotation of 60° about the origin.

Answer:

Let $R = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix}$ and $A = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Then $RA = \begin{pmatrix} 3\cos 60^\circ - 4\sin 60^\circ \\ 3\sin 60^\circ + 4\cos 60^\circ \end{pmatrix} = \begin{pmatrix} -1.964 \\ 4.598 \end{pmatrix}$

and the rotated point has its image at Q = (-1.964, 4.598)

Q35: If $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ find R^{-1}

Rotation of a point

5 min

There is a very useful simulation on the web which allows exploration of these rotations both mathematically and graphically.

Rotation exercise

10 min

There is a web exercise for you to try if you like.

Q36: Find the coordinates of the point (8, 1) rotated through an angle of 30° anticlockwise.

Q37: The point (2, 1) maps to the point (1.537, 1.624) under a rotation about the origin. Find the angle through which the point has been rotated to the nearest degree.

Q38: A point is rotated about the origin by -100° and finishes at the coordinates (-1.796, 1.332). Find the original coordinates to the nearest whole number.

Reflections

It would be useful to find another 2 x 2 matrix which could be used to determine the coordinates of a point reflected in any line through the origin.

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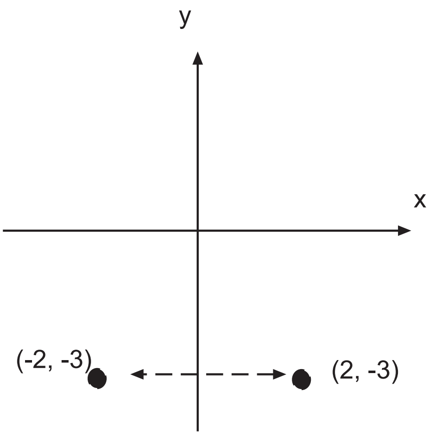
Such a matrix S does exist and SA = B where A is the column matrix representing the original point and B is the matrix representing the reflected coordinates.

Consider the reflection in the y-axis of the point (2, -3)



The matrix S = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ when multiplied to the column matrix $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

representing the point (2, -3) has this effect.

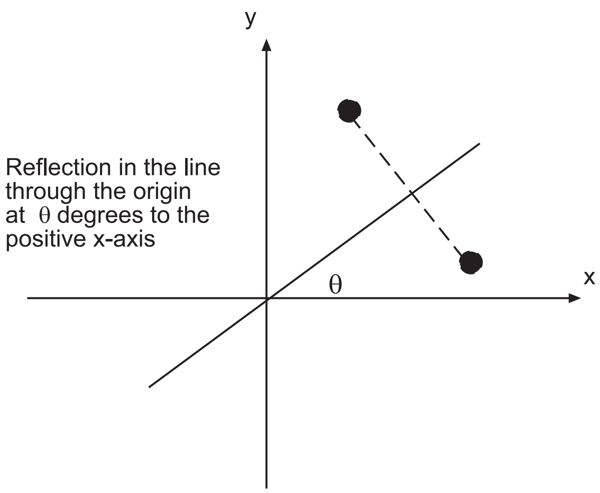
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

Also the matrix S = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acting on the matrix $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ represents a reflection in the x-axis.

The reflection matrix is similar to the rotation matrix but uses the cos and sin of twice the angle θ where θ is the angle that the line of reflection makes with the positive x-axis.

Reflection matrix

The matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ acting, by multiplication on the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, represents the effect of reflection on the point (x, y) in the line through the origin which makes an angle of $\theta°$, -90° $\leq \theta \leq$ 90°



Examples

1. Reflection of a point

Find the coordinates of the point (-2, 5) under a reflection in the line through the origin which makes an angle of 60° with the positive x-axis.

Answer:

Let S = $\begin{pmatrix} \cos 120^\circ & \sin 120^\circ \\ \sin 120^\circ & -\cos 120^\circ \end{pmatrix}$ and A = $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

Then SA = $\begin{pmatrix} -2\cos 120^\circ + 5\sin 120^\circ \\ -2\sin 120^\circ - 5\cos 120^\circ \end{pmatrix} = \begin{pmatrix} 5.330 \\ 0.768 \end{pmatrix}$

and the rotated point has its image at Q = (5.33, 0.768)

2. Find the coordinates of the point (-3, 1) when reflected in the line y = 3x

Answer:

First find the angle which the line makes with the positive x-axis. By simple trigonometry, tan θ = $^1/_3$ and so θ = 18.43°

Let S be the reflection matrix then S = $\begin{pmatrix} \cos 36.86^\circ & \sin 36.86^\circ \\ \sin 36.86^\circ & -\cos 36.86^\circ \end{pmatrix}$

Thus if the point is represented by A = $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, the image of A is

SA = $\begin{pmatrix} \cos 36.86^\circ & \sin 36.86^\circ \\ \sin 36.86^\circ & -\cos 36.86^\circ \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1.8 \\ -2.6 \end{pmatrix}$ to one decimal place.

Reflection of a point

5 min

There is a very useful simulation on the web which allows exploration of these reflections both mathematically and graphically.

Reflection exercise

10 min

There is a web exercise for you to try if you like.

Q39: Find the coordinates, correct to one decimal place, of the point (3, 1) reflected in a line through the origin at an angle of -30 °

Q40: The point (-1, -2) maps to the point (-2, -1) under a reflection in a line at an angle of θ to the x-axis . Find the angle θ to the nearest degree.

Q41: A point is reflected in the line y = - 4x and has its image at the point with coordinates (-1, 0). Find the original coordinates to one decimal place.

Dilatations or scalings

It is clear that for both rotations and reflections the values of the elements in the matrix lie between -1 and 1

However, it is possible to take a point (x, y) and multiply it by a 2 x 2 matrix whose elements are real numbers outwith this range. This produces another range of transformations.

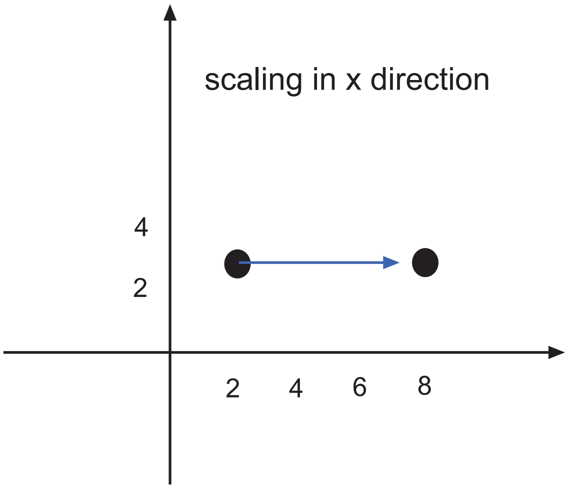
Examples

1. Scaling in the x direction

Consider the effect on the point (2, 3) by multiplying the matrix $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

Answer:

The point has an image at (8, 3). The x-coordinate has been scaled by 4 and the y-coordinate has been left alone (scaled by 1).

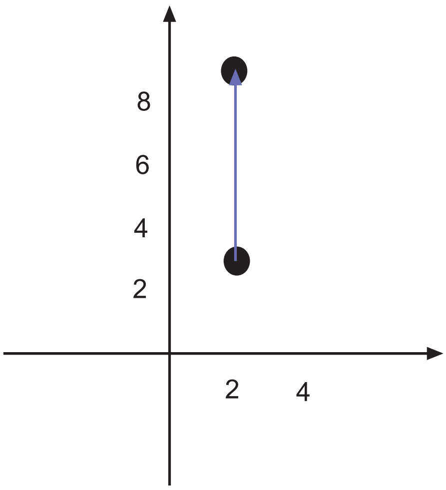


2. Scaling in the y direction

Consider the effect on the point (2, 3) by multiplying the matrix $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Answer:

The point has an image at (2, 9) representing a scaling in the x direction of 1 (no real scaling) and in the y direction by 3



scaling in y direction

3. Scaling in both directions

Consider the effect on the point (2, 3) by multiplying the matrix $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by $\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$

Answer:

$$\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$$

The point is now (8, 15) = (4$\times$ 2, 5 $\times$ 3)

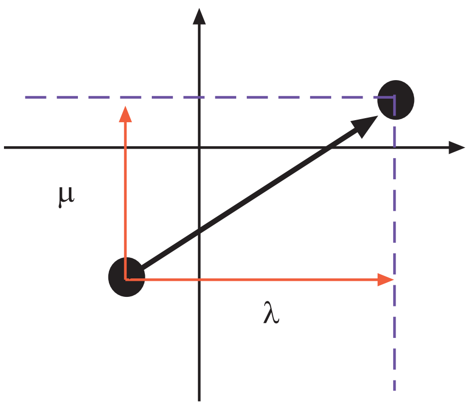
This demonstrates scaling in both the x and y directions: the x direction is scaled by 4 and the y direction by 5

A general definition can now be stated.

Dilatation or scaling matrix

The matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ acting by multiplication on the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ represents the effect on the point (x, y) thus scaling the x-coordinate by λ and the y-coordinate by μ

two way scaling



Scaling of a point

5 min

There is a very useful simulation on the web which allows exploration of these scalings both mathematically and graphically.

Scaling exercise

10 min

There is a web exercise for you to try if you like.

Q42: Find the scaling matrix and the coordinates of the image of the point (3, 1) scaled in the x direction by -3

Q43: The point (-1, -2) maps to the point (-2, -1) under a scaling in both directions. Find the scaling matrix.

Q44: A point (3, 4) is scaled to give an image at the point with coordinates (-1, 0). Find the scaling matrix.

General linear transformations of a line

In a more general way the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will transform sets of points in the x-y plane. In this case a and d are not equal to zero.

However, it is important to note that the origin remains the origin under any linear transformation. Thus a translation is *not* a linear transformation.

Activity

Experiment with some points on the x-y plane using the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d as integers. Check sets of points and verify that the origin remains fixed. This is a good stage at which to experiment with a graphics calculator for these multiplications.

Example : General linear transformation

Find the 2 x 2 matrix which will transform the point (-3, 2) to (-8, -13) and the point (5, 4) to (6, 7)

Answer:

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ -13 \end{pmatrix}$ gives the equations

1) -3a + 2b = -8 and

2) -3c + 2d = -13

Also $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ gives the equations

3) 5a + 4b = 6 and

4) 5c + 4d = 7

Solving the set of equations, 1) and 3) gives a = 2 and b = -1

Similarly the set of equations 2) and 4) gives c = 3 and d = -2

The matrix is $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

General transformation exercise

There is an alternative exercise on the web for you to try if you prefer it.

5 min

Q45: Find the 2 x 2 matrix which will transform the point (-3, 2) to (6, 7) and the point (5, 4) to (-8, -13)

Q46: Find the transformation matrix which takes the point (2, -3) to (-8, 16) and the point (-1, -4) to the point (-7, 3)

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Q47: Find the image of the point (-6, 4) under the transformation by the matrix $\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Q48: Find the transformation matrix which maps the points (-2, 2) to (2, 2) and (-3, 2) to (3, 0)

Q49: Find the 2 x 2 transformation matrix which maps the two points (2, 2) and (3, 4) onto the points (0, 8) and (-2, 13) respectively.

Earlier in this topic orthogonal matrices were briefly mentioned.

Recall the definition: $A^T = A^{-1} \Leftrightarrow$ a matrix is orthogonal. The matrices for rotation and reflection transformations are orthogonal.

Activity

Verify this definition for rotation and reflection matrices using algebra and the trig. property that $\cos^2\theta + \sin^2\theta = 1$

Having established that rotations and reflections have orthogonal matrices, the next step is to look at the transpose and the inverse of each more closely.

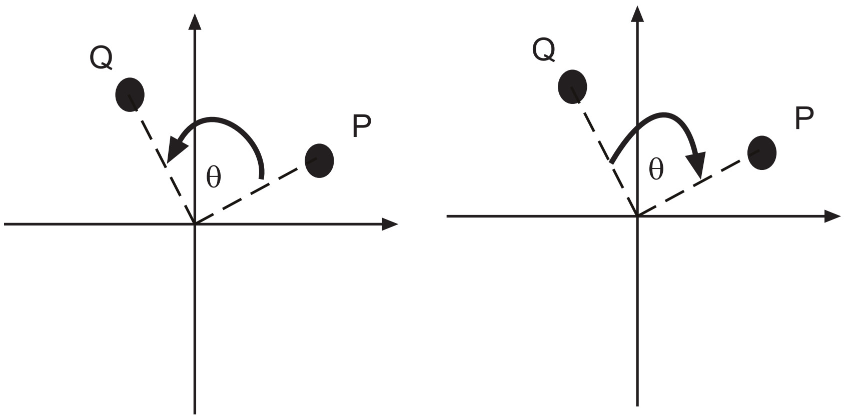
Rotations

Suppose that the point P is transformed under the rotation matrix R by θ ° and maps to the point Q.

The inverse of the rotation matrix R, (R^{-1}) maps the point Q back to the point P. In other words it rotates through an angle of -θ or through θ in a clockwise direction.

Since R is orthogonal, $R^{-1} = R^T$

Thus for any clockwise rotations it is a simple task to find the transpose of the matrix of rotation and multiply the coordinates of the point by this to find the image.



rotation under R rotation under R^T

Reflections

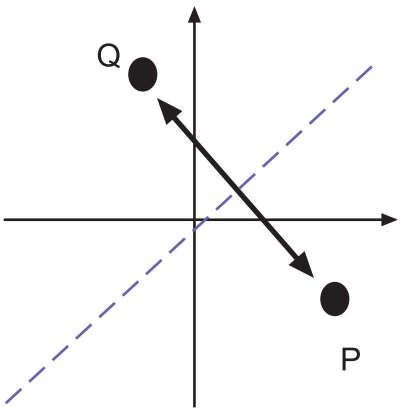
Suppose now that the point P is transformed under the reflection matrix S about a line at θ° through the origin and maps to the point Q.

The inverse of the reflection matrix S, (S^{-1}) will reflect the point Q again in the same line back to the point P.

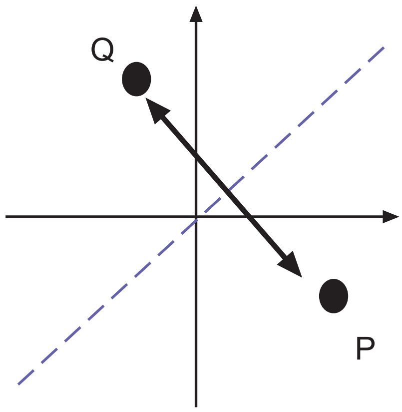
Since S is orthogonal, $S^T = S^{-1}$ and so S^T will reflect Q back to the point P.

In this case, however, the reflection operation is identical, whether it maps P to Q or Q to P and this is confirmed by examination of S^T

For reflections $S^T = S$.



reflection under S



reflection under S^T
(same reflection)

2.7 Summary

The following points and techniques should be know after studying this topic:

- Performing arithmetic operations on two matrices.
- Finding the inverse and determinant of 2 x 2 and 3 x 3 matrices.
- Deducing and using the properties of matrix operations.
- Relating matrix operations to transformations in the (x, y) plane.

2.8 Proofs

Learning Objective

Follow the proofs given and understand the manipulation of the algebra

Proof 1

Property 3: (AB)C = A(BC)

Only the relevant cells are shown to keep this simple. If time permits, try this proof in detail.

Let A = $\begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{pmatrix}$, B = $\begin{pmatrix} b_{11} & ... & b_{1p} \\ ... & ... & ... \\ b_{n1} & ... & b_{np} \end{pmatrix}$ and C = $\begin{pmatrix} c_{11} & ... & c_{1q} \\ ... & ... & ... \\ c_{p1} & ... & c_{pq} \end{pmatrix}$

AB = $\begin{pmatrix} a_{11}b_{11} + ... + a_{1n}b_{n1} & ... & a_{11}b_{1p} + ... + a_{1n}b_{np} \\ ... & ... & ... \\ a_{m1}b_{11} + ... + a_{mn}b_{n1} & ... & a_{m1}b_{1p} + ... + a_{mn}b_{np} \end{pmatrix}$

(AB)C = $\begin{pmatrix} a_{11}b_{11}c_{11} + ... + a_{1n}b_{np}c_{p1} & ... & a_{11}b_{11}c_{1q} + ... + a_{1n}b_{np}c_{pq} \\ ... & ... & ... \\ a_{m1}b_{11}c_{11} + ... + a_{mn}b_{np}c_{p1} & ... & a_{m1}b_{11}c_{1q} + ... + a_{mn}b_{np}c_{pq} \end{pmatrix}$

However BC = $\begin{pmatrix} b_{11}c_{11} + ... + b_{1p}c_{p1} & ... & b_{11}c_{1q} + ... + b_{1p}c_{pq} \\ ... & ... & ... \\ b_{n1}c_{11} + ... + b_{np}c_{p1} & ... & b_{n1}c_{1q} + ... + b_{np}c_{pq} \end{pmatrix}$ and

A(BC) = $\begin{pmatrix} a_{11}b_{11}c_{11} + ... + a_{1n}b_{np}c_{p1} & ... & a_{11}b_{11}c_{1q} + ... + a_{1n}b_{np}c_{pq} \\ ... & ... & ... \\ a_{m1}b_{11}c_{11} + ... + a_{mn}b_{np}c_{p1} & ... & a_{m1}b_{11}c_{1q} + ... + a_{mn}b_{np}c_{pq} \end{pmatrix}$

(AB)C = A(BC)

Proof 2

Property 4: A(B + C) = AB + AC

Let A = $\begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{pmatrix}$, B = $\begin{pmatrix} b_{11} & ... & b_{1p} \\ ... & ... & ... \\ b_{n1} & ... & b_{np} \end{pmatrix}$ and C = $\begin{pmatrix} c_{11} & ... & c_{1p} \\ ... & ... & ... \\ c_{n1} & ... & c_{np} \end{pmatrix}$

B + C = $\begin{pmatrix} b_{11} + c_{11} & ... & b_{1p} + c_{1p} \\ ... & ... & ... \\ b_{n1} + c_{n1} & ... & b_{np} + c_{np} \end{pmatrix}$

and A

(B + C) = $\begin{pmatrix} a_{11}(b_{11} + c_{11}) + ... + a_{1n}(b_{n1} + c_{n1}) & ... & a_{11}(b_{1p} + c_{1p}) + ... + a_{1n}(b_{np} + c_{np}) \\ ... & ... & ... \\ a_{m1}(b_{11} + c_{11}) + ... + a_{mn}(b_{n1} + c_{n1}) & ... & a_{m1}(b_{1p} + c_{1p}) + ... + a_{mn}(b_{np} + c_{np}) \end{pmatrix}$

But AB = $\begin{pmatrix} a_{11}b_{11} + ... + a_{1n}b_{n1} & ... & a_{11}b_{1p} + ... + a_{1n}b_{np} \\ ... & ... & ... \\ a_{m1}b_{11} + ... + a_{mn}b_{n1} & ... & a_{m1}b_{1p} + ... + a_{mn}b_{np} \end{pmatrix}$ and

AC = $\begin{pmatrix} a_{11}c_{11} + ... + a_{1n}c_{n1} & ... & a_{11}c_{1p} + ... + a_{1n}c_{np} \\ ... & ... & ... \\ a_{m1}c_{11} + ... + a_{mn}c_{n1} & ... & a_{m1}c_{1p} + ... + a_{mn}c_{np} \end{pmatrix}$

AB +

AC = $\begin{pmatrix} a_{11}(b_{11} + c_{11}) + ... + a_{1n}(b_{n1} + c_{n1}) & ... & a_{11}(b_{1p} + c_{1p}) + ... + a_{1n}(b_{np} + c_{np}) \\ ... & ... & ... \\ a_{m1}(b_{11} + c_{11}) + ... + a_{mn}(b_{n1} + c_{n1}) & ... & a_{m1}(b_{1p} + c_{1p}) + ... + a_{mn}(b_{np} + c_{np}) \end{pmatrix}$

A(B + C) = AB + AC

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Proof 3

Property 7: $(AB)^T = B^T A^T$

Let $A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & ... & b_{1p} \\ ... & ... & ... \\ b_{n1} & ... & b_{np} \end{pmatrix}$

$(AB) = \begin{pmatrix} a_{11}b_{11} + ... + a_{1n}b_{n1} & ... & a_{11}b_{1p} + ... + a_{1n}b_{np} \\ ... & ... & ... \\ a_{m1}b_{11} + ... + a_{mn}b_{n1} & ... & a_{m1}b_{1p} + ... + a_{mn}b_{np} \end{pmatrix}$ and so

$(AB)^T = \begin{pmatrix} a_{11}b_{11} + ... + a_{1n}b_{n1} & ... & a_{m1}b_{11} + ... + a_{mn}b_{n1} \\ ... & ... & ... \\ a_{11}b_{1p} + ... + a_{1n}b_{np} & ... & a_{m1}b_{1p} + ... + a_{mn}b_{np} \end{pmatrix}$

But $B^T = \begin{pmatrix} b_{11} & ... & b_{n1} \\ ... & ... & ... \\ b_{1p} & ... & b_{np} \end{pmatrix}$ and $A^T = \begin{pmatrix} a_{11} & ... & a_{m1} \\ ... & ... & ... \\ a_{1n} & ... & a_{mn} \end{pmatrix}$

Thus $B^T A^T = \begin{pmatrix} a_{11}b_{11} + ... + a_{1n}b_{n1} & ... & a_{m1}b_{11} + ... + a_{mn}b_{n1} \\ ... & ... & ... \\ a_{11}b_{1p} + ... + a_{1n}b_{np} & ... & a_{m1}b_{1p} + ... + a_{mn}b_{np} \end{pmatrix}$ since ab = ba in real numbers.

$(AB)^T = B^T A^T$

2.9 Extended information

Learning Objective

Show an awareness of additional information on this topic

SEKI KOWA

He was a Japanese mathematician who used matrix methods taken from the Chinese ideas of the centuries before.

LEIBNITZ

He was working on matrices at the same time as Seki Kowa using determinants applied to systems of three equations in three unknowns.

LAPLACE

The method of expanding the determinant shown in this course is named after him.

CAUCHY

The word determinant was first used as we know it by Cauchy.

SYLVESTER

The term matrix was not used until 1850 when the English mathematician defined it as an array.

CAYLEY

In the late 1800s Cayley identified many of the algebraic properties of matrices which are studied today.

2.10 Review exercise

Review exercise

15 min

There is a web version of this review exercise for you to try if you prefer it.

Q50: Let A = $\begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$, B = $\begin{pmatrix} -4 & 3 \\ -5 & -1 \end{pmatrix}$,

C = $\begin{pmatrix} 2 & -2 \\ 4 & 3 \end{pmatrix}$ and D = $\begin{pmatrix} -1 & 0 & -2 \\ 3 & -4 & 2 \\ -3 & 2 & 5 \end{pmatrix}$

Find

a) BC
b) A^{-1}
c) A -3B + C
d) det D

Q51: Let A = $\begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}$, B = $\begin{pmatrix} 0 & -2 \\ -3 & -5 \end{pmatrix}$,

C = $\begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$ and D = $\begin{pmatrix} 1 & 3 & 3 \\ 3 & -3 & 2 \\ 0 & -1 & 2 \end{pmatrix}$

Find

a) BC
b) A^{-1}
c) A +2B - C
d) det D

2.11 Advanced review exercise

Advanced review exercise

15 min

There is another exercise on the web that you may like to try.

Q52: The n x n matrix A satisfies the equation $2A^2 = A + I$ where I is the n x n identity matrix. Show that A is invertible and express A^{-1} in the form pA + qI.

Q53: If A is a square matrix which satisfies the equation $A^2 - 4I = 0$ find two solutions for A.

Q54: The point (2, -3) is transformed first by a rotation through -60° and then by a scaling of -2 in the x direction and 3 in the y direction. Finally the point is further transformed by a reflection in the x-axis. Give the coordinates of the image of the point after these transformations have taken place to one decimal place.

Q55: If A is an orthogonal matrix show that the matrix $(AA^T)^{-1} + A(3A^T - A^{-1})$ is 3I where I is the 2 x 2 identity matrix.

2.12 Set review exercise

Set review exercise

10 min

The answers for this exercise are only available on the web by entering the answers obtained in an exercise called 'set review exercise'. The questions may be structured differently but will require the same answers.

Q56: Let $A = \begin{pmatrix} -2 & 0 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix}$,

$C = \begin{pmatrix} -5 & -1 \\ 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}$

Find

a) BC
b) A^{-1}
c) 2A - B + C
d) det D

Q57: Let $A = \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ -5 & -1 \end{pmatrix}$,

$C = \begin{pmatrix} 2 & 3 \\ -2 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 3 & 1 \\ -2 & 4 & 1 \\ 3 & 4 & 1 \end{pmatrix}$

Find

a) AB
b) C^{-1}
c) 2A - B - 3C
d) det D

Topic 3

Further sequences and series

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Learning Objectives

Understand and use further aspects of sequences and series.

Minimum performance criteria:

- *Use the Maclaurin series expansion to find power series for simple functions to a stated number of terms.*
- *Find a solution of a simple non-linear equation, to a prescribed degree of accuracy, using an iterative scheme of the form $x_{n+1} = g(x_n)$ where $x = g(x)$ is a rearrangement of the original equation.*

Prerequisites

Before attempting this unit you should have a thorough knowledge of the following:

- Geometric series, in particular the formula for the sum to infinity of a geometric series.
- The Binomial expansion for an expression in the form $(a + x)^{-1}$
- Higher derivatives of simple functions.
- Linear recurrence relations.

10 min

Revision Exercise

Q1: Find the sum to infinity for the geometric series

$16 + 8 + 4 + 2 + 1 + {}^1/_2 + ...$

Q2: Write down the binomial expansion for $(1 + x)^{-1}$, $-1 < x < 1$

Q3: For $f(x) = e^{2x}$ calculate the derivatives

$f'(x)$, $f''(x)$ and $f'''(x)$

Q4: Find the first five terms and the limit for the recurrence relation

$u_{n+1} = 0.7u_n + 9$, $u_0 = 5$

3.1 Maclaurin series

3.1.1 Power series

Learning Objective

Recognise power series

A **power series** is an expression of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n + ...$$

where $a_0, a_1, a_2, a_3, ..., a_n, ...$ are constants and x is a variable. It is called a power series as it is made up of a sequence of powers of x with coefficients $a_0, a_1, a_2, a_3, ..., a_n, ...$

Power series are useful in solving differential equations that occur in physics, including the equations that describe motion of a simple pendulum, vibrating strings, heat flow, electrical current, and many other examples.

In **numerical analysis**, power series can be used to determine how many decimal places are required in a computation to guarantee a specified accuracy.

Numerical analysis is a branch of mathematics. It can be described as the analysis and solution of problems which require calculation.

It is also useful if we are able to express some simple functions such as e^x, sin x, cos x,

$\tan^{-1}x$, $(1 + x)^n$ and ln (1 + x) in terms of power series.

Examples

1. $1 + x + x^2 + x^3 + x^4 + ... x^n + ...$ is an example of a power series with $1 = a_0 = a_1 = a_2 = a_3 = a_4 = ... = a_n = ...$

You should also recognise this as a geometric series.

2. Another important example of a power series is

$\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$

This time $a_r = \frac{1}{r!}$

This power series converges for all values of $x \in \mathbb{R}$

Q5: Substitute x = 1 into the above power series and calculate the sum of the first ten terms, S_{10}, to 6 decimal places. Do you recognise this answer?

In this topic we will learn a method that allows us to rewrite a variety of functions as power series.

3.1.2 Maclaurin series for simple functions

Learning Objective

Generate Maclaurin series for various functions using the given formula

Suppose we have a function f (x), which we are able to keep differentiating as often as we want and that there is no problem differentiating when x = 0. Functions like this do exist and for example e^x and sin x are functions that satisfy these conditions. With these conditions we are able to find a special type of power series called the Maclaurin series.

The Maclaurin series generated by the function f (x) is

$$\sum_{r=0}^{\infty} f^{(r)}(0)\frac{x^r}{r!} = f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + ... + f^{(n)}(0)\frac{x^n}{n!} + ...$$

The following examples show how we can find the Maclaurin Series generated by e^x and sin x

Example Find the Maclaurin series generated by $f(x) = e^x$

Answer

When $f(x) = e^x$ is repeatedly differentiated we obtain

$$\begin{aligned} f(x) &= e^x & f(0) &= e^0 = 1\\ f^{(1)}(x &= e^x & f^{(1)}(0) &= e^0 = 1\\ f^{(2)}(x) &= e^x & f^{(2)}(0) &= e^0 = 1\\ f^{(3)}(x) &= e^x & f^{(3)}(0) &= e^0 = 1\\ f^{(4)}(x &= e^x & f^{(4)}(0) &= e^0 = 1\\ f^{(5)}(x) &= e^x & f^{(5)}(0) &= e^0 = 1 \end{aligned}$$

Therefore the Maclaurin series generated by $f(x) = e^x$ becomes

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0) &= f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + f^{(5)}(0)\frac{x^5}{5!} + \ldots\\ &= 1 + (1)\frac{x}{1!} + (1)\frac{x^2}{2!} + (1)\frac{x^3}{3!} + (1)\frac{x^4}{4!} + (1)\frac{x^5}{5!}\ldots\\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \end{aligned}$$

You should recognise this from before.

Example Find the Maclaurin series generated by $f(x) = \sin x$

Answer

When $f(x) = \sin x$ is repeatedly differentiated we obtain

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin(0) = 0\\ f^{(1)}(x) &= \cos x & f^{(1)}(0) &= \cos(0) = 1\\ f^{(2)}(x) &= -\sin x & f^{(2)}(0) &= -\sin(0) = 0\\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -\cos(0) = -1\\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin(0) = 0\\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos(0) = 1 \end{aligned}$$

Therefore the Maclaurin series generated by $f(x) = \sin x$ becomes

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0) &= f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + f^{(5)}(0)\frac{x^5}{5!}\ldots\\ &= 0 + (1)\frac{x}{1!} + (0)\frac{x^2}{2!} + (-1)\frac{x^3}{3!} + (0)\frac{x^4}{4!} + (1)\frac{x^5}{5!}\ldots\\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots \end{aligned}$$

Now try the questions in Exercise 1.

Exercise 1

An on-line assessment is available at this point, which you might find helpful.

15 min

Q6: Find the Maclaurin series generated by the following functions

a) $f(x) = \cos x$

b) $f(x) = (1 + x)^n$

c) $f(x) = \ln(1 + x)$

3.1.3 Maclaurin's theorem

Learning Objective

Appreciate that given certain conditions the Maclaurin series generated by a function is actually equal to that function

What is the relationship between the Maclaurin series of a function and the function itself?

Suppose we have a function $f(x)$ which we are able to keep differentiating as often as we want and that there is no problem differentiating when $x = 0$. Also suppose that it is possible to write this function as a series expansion so that

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ...$

Then the following definition applies

Maclaurin's theorem

Maclaurin's theorem states that

$$f(x) = \sum_{r=0}^{\infty} f^{(r)}(0)\frac{x^r}{r!}$$

$$= f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + ... + f^{(n)}(0)\frac{x^n}{n!} + ...$$

Notice that this theorem claims that the function $f(x)$ is actually equal to its infinite power series expansion. The theorem is named after the Scottish mathematician Colin Maclaurin (1698-1746) who first proposed this result in his publication *Treatise of fluxions*.

We can see why this is true from the following reasoning.

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ...$

If we substitute $x = 0$ in the expansion we get $f(0) = a_0$

Differentiating with respect to x, we obtain

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + ...$

Now substituting $x = 0$ into this equation gives us $f'(0) = a_1 = 1!a_1$

We can repeat this process again.

Differentiating again with respect to x, we obtain

$f''(x) = (2 \times 1)a_2 + (3 \times 2)a_3x + (4 \times 3)a_4x^2 + (5 \times 4)a_5x^3 + ...$

Therefore $f''(0) = (2 \times 1)a_2 = 2!a_2$

Differentiating once more with respect to x, we obtain

$f'''(x) = (2 \times 1)a_2 + (3 \times 2 \times 1)a_3 + (4 \times 3 \times 2)a_4x + (5 \times 4 \times 3)a_5x^2 + ...$

Therefore $f'''(0) = (3 \times 2 \times 1)a_3 = 3!a_3$

Remember that for higher derivatives it is often more convenient to replace a series of dashes with a number.

In other words, for example, f''' can be rewritten as $f^{(3)}$

Q7: Do the next differentation yourself to find $f^{(4)}(0)$

Rearranging the previous results we can rewrite the coefficients as follows,

$a_0 = f(0)$

$a_1 = \dfrac{f^{(1)}(0)}{1!}$

$a_2 = \dfrac{f^{(2)}(0)}{2!}$

$a_3 = \dfrac{f^{(3)}(0)}{3!}$

$a_4 = \dfrac{f^{(4)}(0)}{4!}$

Since we assumed that we could keep on differentiating f (x) then we can also say that

$a_n = \frac{f^{(n)}(0)}{n!}$

Now substituting these expressions for a_r back into the power series

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5...$

gives the Maclaurin series expansion.

$f(x) = f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + ...$

Note that this result depends on us being able to differentiate the infinite series term-by-term and is only valid within an interval of convergence.

The following example should help you understand more clearly what we mean by this.

Example Consider the power series

$1 + x + x^2 + x^3 + x^4 + ... + x^n + ...$

In Unit 2, Sequences and Series, we learned that for a geometric **convergent series**

$S_\infty = a + ar + ar^2 + ar^3 + ar^4 + ... = \frac{a}{1 - r}$ where $-1 < r < 1$

The interval $-1 < r < 1$ is known as the interval of convergence for the previous series. For values of r outside this interval the series does not converge and we say that the series is a **divergent series** for $r \leq -1$ and for $r \geq 1$

divergent series

A **divergent series** is one which is not convergent. For example 1 + 2 + 3 + 4 + 5 ... is a divergent series. The sum of this series will continue increasing the more terms we add. It will not tend towards a limit.

Therefore comparing the power series

$1 + x + x^2 + x^3 + x^4 + ... + x^n + ...$

to a geometric series we can see that this series is convergent for $-1 < x < 1$ and from the formula for S_∞, taking a = 1 and r = x we can write

$1 + x + x^2 + x^3 + x^4 + ... + x^n + ... = {}^1/_{1 - x}$, for $-1 < x < 1$

$-1 < x < 1$ is the interval of convergence for this series. Outside the interval of convergence the power series is divergent.

For example, when x = 1 the series is simply 1 + 1 + 1 + 1 + ... + 1 + ... which is obviously divergent as the sum of the series will continue increasing the more terms we add. It will not tend towards a limit.

convergent series

A **convergent series** is one for which the limit of partial sums exists. This limit is called the sum and is denoted by S_∞

It was also established in Unit 2, Sequences and Series, that we can perform a binomial expansion on ${}^1/_{1 - x}$ which of course can be written as $(1 - x)^{-1}$. Remember that when the binomial expansion is used with negative powers the expansion is infinite and to ensure that the infinite series converges we need $-1 < x < 1$, just as in the geometric series formula.

The Binomial expansion gives

$$\begin{aligned}(1 - x)^{-1} &= 1 + \frac{(-1)}{1!}(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 \\ &\quad + \frac{(-1)(-2)(-3)(-4)}{4!}(-x)^4 + ... \\ &= 1 + x + x^2 + x^3 + x^4 + ...\end{aligned}$$

which you will notice gives the same series as previously.

It will be useful for you to remember the following information.

> We can normally expect a Maclaurin series to converge to its generating function in an interval about the origin. However, for many functions this is the entire x-axis.
>
> For example, the Maclaurin series for functions such as sin x, cos x, e^x all converge for $x \in \mathbb{R}$
>
> Whereas the Maclaurin series for
>
> $\ln(1 + x)$ converges for $-1 < x \leqslant 1$
>
> $\dfrac{1}{1 + x}$ converges for $|x| < 1$
>
> $\tan^{-1}x$ converges for $|x| \leqslant 1$

10 min

The Maclaurin series for sin x

There is a simulation on the web that allows you to compare the graph for sin x with the graphs for partial sums of its Maclaurin series.

3.1.4 The Maclaurin series for $\tan^{-1}x$

We can find the Maclaurin series for $\tan^{-1}x$ in the same way as for the previous functions. However the working soon becomes a bit complicated as you can see here.

When f (x) = $\tan^{-1}x$ is repeatedly differentiated we obtain

$$f(x) = \tan^{-1}x \qquad f(0) = 0$$

$$f^{(1)}(x) = \frac{1}{1+x^2} \qquad f^{(1)}(0) = 1$$

$$f^{(2)}(x) = \frac{-2x}{(1+x^2)^2} \qquad f^{(2)}(0) = 0$$

$$f^{(3)}(x) = \frac{6x^2-2}{(1+x^2)^3} \qquad f^{(3)}(0) = -2$$

From this we only obtain the first two terms in the Maclaurin series.

$$\begin{aligned}
\tan^{-1}x &= \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0) \\
&= f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + f^{(5)}(0)\frac{x^5}{5!}\ldots \\
&= 0 + (1)\frac{x}{1!} + (0)\frac{x^2}{2!} + (-2)\frac{x^3}{3!} + \ldots \\
&= x - \frac{x^3}{3} + \ldots
\end{aligned}$$

You can see that to differentiate further and so obtain more terms in the series soon becomes quite tedious. The following method gives another way to obtain more terms.

$$f(x) = \tan^{-1}x$$

$$f'(x) = \frac{1}{1+x^2} = \left(1+x^2\right)^{-1}$$

Provided $|x^2| < 1$ we can now use the binomial expansion to give

$$\begin{aligned}
f'(x) = {} & 1 + \frac{(-1)}{1!}x^2 + \frac{(-1)(-2)}{2!}\left(x^2\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(x^2\right)^3 \\
& + \frac{(-1)(-2)(-3)(-4)}{4!}\left(x^2\right)^4 + \ldots \\
= {} & 1 - x^2 + x^4 - x^6 + x^8 - \ldots
\end{aligned}$$

Integrating this will take us back to f (x) = $\tan^{-1}x$

$$\tan^{-1}x = C + x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 \ldots$$

C is the integrating factor. However when x = 0, $\tan^{-1}x = 0$ and therefore C = 0

So we can now write

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 \ldots$$

The previous series for $\tan^{-1}x$ actually converges for $|x| \leq 1$ and we can use it to obtain an approximation for π in the following way.

Let x = 1 then

$\tan^{-1}1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$

and so

$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)$

The previous series is known as the Leibniz formula for π. However, this series converges very slowly so in practice it is not used in approximating π to many decimal places. In fact 1000 terms are needed before it gives a value accurate to 4 decimal places.

There is more information about different methods for calculating π in the extended information chapter.

What happens when you try to obtain the Maclaurin series for the following functions?

Q8: f (x) = ln x

Q9: f (x) = $\sqrt{x}$

Q10: f (x) = cot x

3.1.5 Further Maclaurin series

Learning Objective

Calculate the Maclaurin series for simple composite functions

It is also possible to find Maclaurin series for functions such as e^{-3x} and sin 2x. The following example shows how this can be done.

Example Find the Maclaurin series for e^{-3x}

Answer

When f (x) = e^{-3x} is repeatedly differentiated we obtain

$f(x) = e^{-3x}$	$f(0) = e^0 = 1$
$f^{(1)}(x) = -3e^{-3x}$	$f^{(1)}(0) = -3e^0 = -3$
$f^{(2)}(x) = (-3)^2e^{-3x}$	$f^{(2)}(x) = (-3)^2e^0 = (-3)^2$
$f^{(3)}(x) = (-3)^3e^{-3x}$	$f^{(3)}(x) = (-3)^3e^0 = (-3)^3$
$f^{(4)}(x) = (-3)^4e^{-3x}$	$f^{(4)}(x) = (-3)^4e^0 = (-3)^4$
$f^{(5)}(x) = (-3)^5e^{-3x}$	$f^{(5)}(x) = (-3)^5e^0 = (-3)^5$

Therefore the Maclaurin series for f (x) = e^{-3x} becomes

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$$
\begin{aligned}
e^{-3x} &= \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0) \\
&= f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \frac{x^4}{4!} f^{(4)}(0) + \frac{x^5}{5!} f^{(5)}(0) \ldots \\
&= 1 + \frac{x}{1!}(-3) + \frac{x^2}{2!}(-3)^2 + \frac{x^3}{3!}(-3)^3 + \frac{x^4}{4!}(-3)^4 + \frac{x^5}{5!}(-3)^5 \ldots \\
&= 1 - \frac{3x}{1!} + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} - \frac{(3x)^5}{5!} + \ldots
\end{aligned}
$$

We could have obtained this same result if we had substituted (-3x) for x in the Maclaurin series for e^x. Check this for yourself!

Now try the questions in Exercise 2.

Exercise 2

30 min

An on-line assessment is available at this point, which you might find helpful.

Q11: Find the Maclaurin series for the following functions

a) e^{5x}, $x \in \mathbb{R}$

b) sin 2x, $x \in \mathbb{R}$

c) cos 3x, $x \in \mathbb{R}$

d) cos (-x), $x \in \mathbb{R}$

e) ln (1 - 3x), $-{}^{1}/_{3} < x < {}^{1}/_{3}$

Q12:

a) For $i = \sqrt{-1}$ write down the Maclaurin expansion for e^{ix}

b) Can you suggest a connection between e^{ix}, sin x and cos x

3.1.6 Maclaurin series expansion to a given number of terms

Learning Objective

Determine the Maclaurin series expansion for a given function to a specified number of terms

Often you may be required to find a specific number of terms in a Maclaurin series expansion. We proceed as before as you can see in the following example.

Examples

1. Use Maclaurin's theorem to write down the expansion of $(1 + x)^{-3}$ as far as the term in x^3

Answer

When $(1 + x)^{-3}$ is repeatedly differentiated we obtain

$f(x) = (1 + x)^{-3}$	$f(0) = 1$
$f^{(1)}(x) = -3(1 + x)^{-4}$	$f^{(1)}(0) = -3$
$f^{(2)}(x) = (-3)(-4)(1 + x)^{-5}$	$f^{(2)}(0) = (-3)(-4)$
$f^{(3)}(x) = (-3)(-4)(-5)(1 + x)^{-6}$	$f^{(3)}(0) = (-3)(-4)(-5)$

We may stop here as we are only required to find the series as far as the term in x^3

From this we obtain the Maclaurin series

$$
\begin{aligned}
(1 + x)^{-3} &= \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0) \\
&= f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \ldots \\
&= 1 + \frac{x}{1!}(-3) + \frac{x^2}{2!}(-3)(-4) + \frac{x^3}{3!}(-3)(-4)(-5) + \ldots \\
&= 1 - 3x + 6x^2 - 10x^3 + \ldots
\end{aligned}
$$

Thus the Maclaurin series for $(1 + x)^{-3}$, as far as the term in x^3, is $1 - 3x + 6x^2 - 10x^3$

This is also known as the Maclaurin series to third order because the highest power of x in the expansion is 3

2. Be careful if you are asked to find the Maclaurin series generated by, for example, $(1 + x)^5$. You should notice that since this expression has a positive integer power then it will have a finite number of terms when expanded.

Answer

When $(1 + x)^5$ is repeatedly differentiated we obtain

$f(x) = (1 + x)^5$	$f(0) = 1$
$f^{(1)}(x) = 5(1 + x)^4$	$f^{(1)}(0) = 5$
$f^{(2)}(x) = (5)(4)(1 + x)^3$	$f^{(2)}(0) = 20$
$f^{(3)}(x) = (5)(4)(3)(1 + x)^2$	$f^{(3)}(0) = 60$
$f^{(4)}(x) = (5)(4)(3)(2)(1 + x)^1$	$f^{(4)}(0) = 120$
$f^{(5)}(x) = (5)(4)(3)(2)$	$f^{(5)}(0) = 120$
$f^{(6)}(x)$ and further derivatives = 0	$f^{(6)}(0)$ and further derivatives = 0

From this we obtain the following Maclaurin series

$$
\begin{aligned}
(1 + x)^5 &= \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0) \\
&= f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \frac{x^4}{3!} f^{(4)}(0) + \frac{x^5}{3!} f^{(5)}(0) \\
&= 1 + \frac{x}{1!}(5) + \frac{x^2}{2!}(20) + \frac{x^3}{3!}(60) + \frac{x^4}{4!}(120) + \frac{x^5}{5!}(120) \\
&= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5
\end{aligned}
$$

You should recognise that this is the same as the binomial expansion for $(1 + x)^5$

Now try the questions in Exercise 3.

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30 min

Exercise 3

An on-line assessment is available at this point, which you might find helpful.

Q13: Use Maclaurin's theorem to write down the expansions, as far as the term in x^4, of

a) $\sqrt{(1 - x)}$, $|x| < 1$

b) $(1 + x)^{-5}$, $|x| < 1$

c) $(x + 1)^{3/2}$, $|x| < 1$

Q14:

Find all the terms in the Maclaurin series generated by

a) $(2 + x)^4$

b) $(1 - 2x)^3$

Q15:

Use Maclaurin's theorem to write down the expansions, as far as the term in x^3, of

a) $\frac{1}{2x + 3}$, $|x| < \frac{1}{2}$

b) $\frac{1}{(x - 2)^3}$, $|x| < 1$

3.2 Iterative Schemes

Iteration

Iteration is the successive repetition of a mathematical process using the result of one stage as the input for the next.

In this section we use iteration to solve equations.

3.2.1 Recurrence relations

Learning Objective

Investigate when a first order linear recurrence relation converges to a limit

You will already be familiar with the concept of a recurrence relation. However, as a reminder here is the definition.

recurrence relation

A **recurrence relation** describes a sequence of terms u_0, u_1, u_2, u_3, ..., u_n, u_{n+1}, ... where each term is a function of previous terms.

So in this way a recurrence relation is a type of iterative process.

There are various types of recurrence relation.

first order recurrence relation

A **first order recurrence relation** is a recurrence relation where u_{n+1} depends only on u_n and not on values of u_r where $r < n$

Example $u_{n+1} = 2u_n - 4$ is an example of a **first order recurrence relation.**

second order recurrence relation A **second order recurrence relation** is a recurrence relation of the form $u_{n+2} = au_{n+1} + bu_n + c$, with a and b $\neq 0$. Note that u_{n+2} depends on the previous two terms in the sequence.

Example $u_{n+2} = 2u_{n+1} + 0.5u_n - 6$ is a **second order recurrence relation**. Note that u_{n+2} depends on the previous two terms in the sequence.

Linear first order recurrence relation

A **linear first order recurrence relation** is a recurrence relation of the form
$u_{n+1} = au_n + b,\ a \neq 0$

Example $u_{n+1} = 0.8u_n + 2$ is a linear first order recurrence relation.

Example

However, $u_{n+1} = \ln(u_n) + 3$ is a **non-linear** first order recurrence relation. It includes the non-linear term $\ln(u_n)$

Previous work has concentrated on first order linear recurrence relations. It will be useful if we are reminded of some of the properties of these recurrence relations. In particular it is important to be able to decide from a given recurrence relation whether the sequence of terms will converge or diverge. Look at these examples.

Examples

1. For the recurrence relation $u_{n+1} = 0.2u_n + 4$ with $u_0 = 1$ we get the following sequence of terms.

1, 4.2, 4.84, 4.968, 4.9936, 4.99872, 4.999744, ...

This sequence of terms **converges** to the limit 5

2. For the recurrence relation $u_{n+1} = 3u_n + 2$ with $u_0 = 1$ we get the following sequence of terms.

5, 17, 53, 161, 485, 1457, 4373, 13121, ...

This sequence continues to get bigger and bigger. It **diverges**.

Now try the following questions in Exercise 4.

20 min

Exercise 4

Q16:

Consider the recurrence relation $u_{n+1} = au_n + 4$. Investigate whether this recurrence relation converges or diverges for the following values of a

a) 4

b) 1

c) 0.8

d) 0.5

e) 0.2

f) 0

g) -0.2

h) -0.6

i) -2

Q17:

For which values of a would you expect the recurrence relation to converge to a limit?

Q18: Consider the recurrence relation $u_{n+1} = 0.2u_n + b$ with $u_0 = 0$. Investigate the effect of substituting b with the following values.

a) 10

b) 0.6

c) -0.2

d) -4

Q19:

What effect does changing b seem to have on the recurrence relation?

Q20: Consider the recurrence relation $u_{n+1} = 0.2u_n + 6$

Investigate the effect of substituting u_0 with the following values.

a) 0

b) 2

c) 10

d) 100

Q21:

What effect does changing u_0 seem to have on the convergence of the recurrence relation?

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3.2.2 Recurrence relations and corresponding equations

Learning Objective

Equate recurrence relations with their corresponding equations and identify when the recurrence relation converges to a fixed point

In the previous section we were able to confirm that

> For the linear recurrence relation $u_{n+1} = au_n + b$, u_n tends to a limit, L,
>
> when $-1 < a < 1$
>
> L is given by the formula $L = \frac{b}{1 - a}$

We can derive the above formula for L from the following.

In general we can say that if $u_{n+1} \Rightarrow L$ as $n \Rightarrow \infty$ then we also have that $u_n \Rightarrow L$ as $n \Rightarrow \infty$

Therefore, as $n \Rightarrow \infty$, the formula $u_{n+1} = au_n + b$ tends to

$$\begin{aligned} L &= aL + b \\ L - aL &= b \\ L(1 - a) &= b \\ L &= \frac{b}{1 - a} \end{aligned}$$

Now consider the following.

Q22: Solve the following equations

a) $x = 4x + 4$

b) $x = 0.8x + 4$

c) $x = -0.2x + 4$

Compare your answers to these equations to the following recurrence relations which were already investigated in Exercise 4.

$u_{n+1} = 4u_n + 4$ (Ex 4 Q16 a)

$u_{n+1} = 0.8u_n + 4$ (Ex 4 Q16 c)

$u_{n+1} = -0.2u_n + 4$ (Ex 4 Q 16g)

Is there a connection?

You should notice that the above recurrence relations have corresponding equations. So, for example, $u_{n+1} = 0.8u_n + 4$ has corresponding equation $x = 0.8x + 4$ When the recurrence relation converges ($-1 < a < 1$) then the limit of the recurrence relation is equal to the solution of the corresponding equation. This solution is also known as the **fixed point** of the recurrence relation. Divergent recurrence relations also have a fixed point but application of the recurrence relation does not provide the fixed point. You will understand this more clearly once you have done the exercise on Staircase and Cobweb diagrams later in this section.

fixed point

The recurrence relation $u_{n+1} = au_n + b$ has **fixed point** given by $^b/_{(1 - a)}$. The recurrence relation will only converge to this fixed point if $(-1 < a < 1)$

Q23: Write down a recurrence relation which corresponds to each of the following equations.

a) x = -0.3x + 7

b) x = 2x = 5

c) x = 0.4x + 6

d) 0.2x = 4

e) 1.2x = 3.6

f) -0.5x = 15

Q24: Which of these recurrence relations converge to a limit that is the same as the solution of the equation?

The following diagrams may help you to understand why this happens.

Staircase and Cobweb Diagrams

15 min

You will find interactive versions of the following Staircase and Cobweb diagrams on the course web site.

The recurrence relation $x_{n+1} = ax_n + b$ with initial value x_0 can be represented geometrically by the straight lines y = x and y = ax + b

The strategy given here can then be used to draw the diagrams.

1. Draw the lines y = x and y = ax + b. From the given value of x^0 draw a vertical line to meet the line y = ax + b. The coordinates of the point of intersection are (x_n, x_{n+1}) with $x_{n+1} = ax_n + b$ and n = 0
2. Draw a horizontal line from (x_n, x_{n+1}) to meet the line y = x. The point of intersection has coordinates (x_{n+1}, x_{n+1})
3. Draw a vertical line from (x_{n+1}, x_{n+1}) to meet the line y = ax + b. The point of intersection has coordinates (x_{n+1}, x_{n+2}) with $x_{n+2} = ax_{n+1} + b$
4. Repeat steps 2, 3 and 4 of this iterative process.

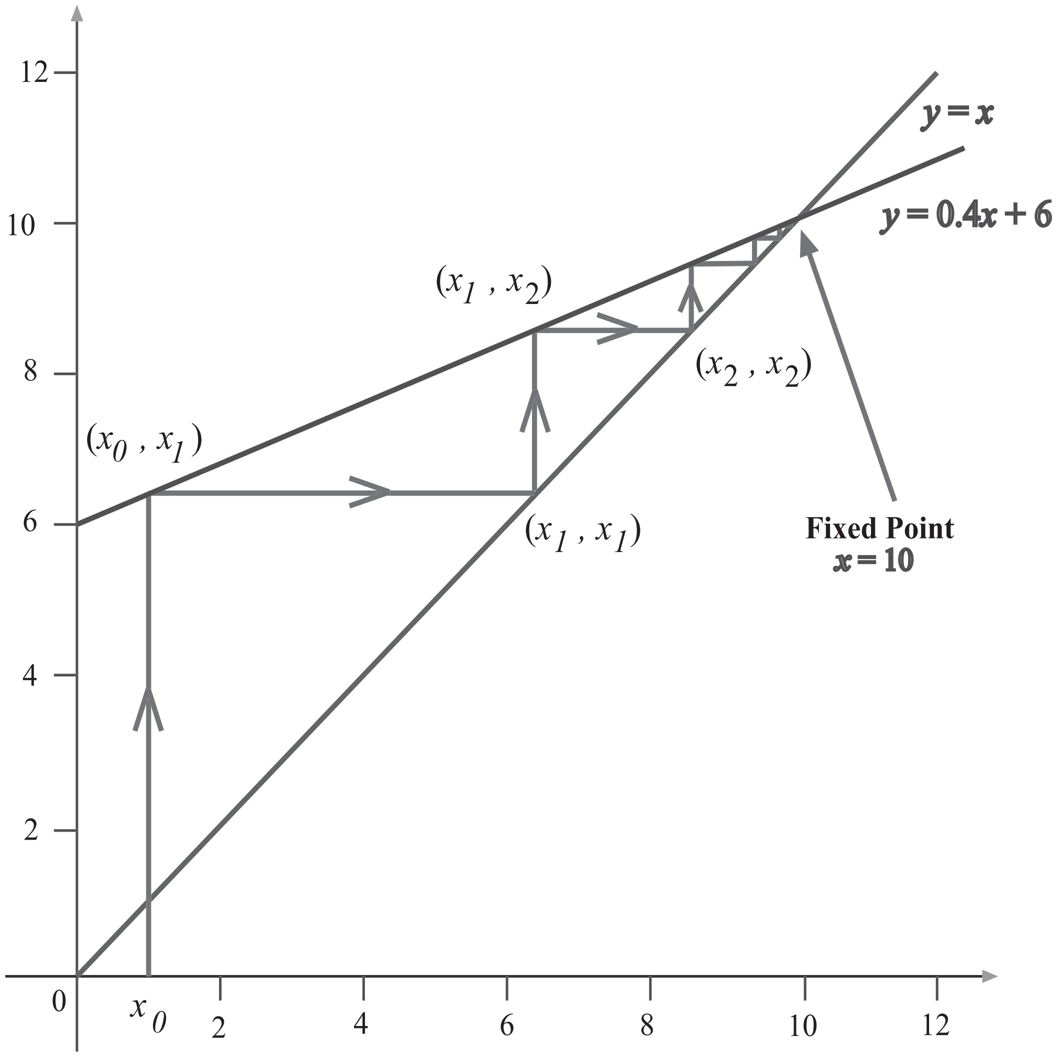
This single process gives two types of diagram.

- A staircase diagram when $a > 0$
- A cobweb diagram when $a < 0$

There follows examples of these two types of diagram.

Example : Staircase Diagram

(a $>$ 0)



This diagram shows the straight lines y = 0.4x + 6 and y = x which is a geometrical illustration of the recurrence relation $u_{n+1} = 0.4u_n + 6$

This iterative process can be written in x notation as $x_{n+1} = 0.4x_n + 6$

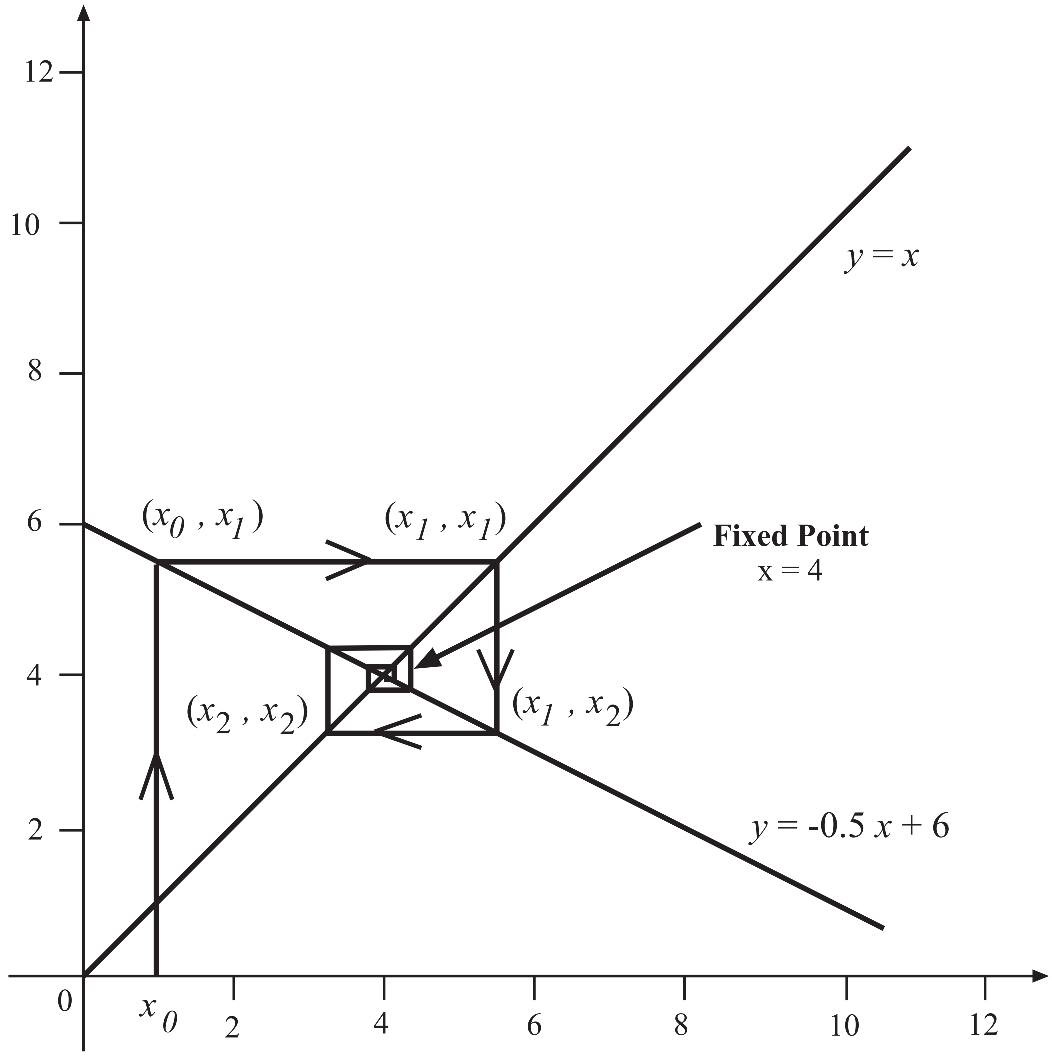
The initial value in this example is $x_0 = 1$

The iterative process is repeated and the value of x_n tends to the fixed point of the recurrence relation as n $\Rightarrow \infty$. This is also the solution to the equation x = 0.4x + 6 which is x = 10

Notice that the fixed point is where the lines y = x and y = 0.4x + 6 intersect.

Example : Cobweb Diagram

(a $<$ 0)



This diagram shows the straight lines y = -0.5x + 6 and y = x which is a geometrical illustration of the recurrence relation $u_{n+1} = -0.5u_n + 6$

This iterative process can be written in x notation as $x_{n+1} = -0.5x_n + 6$

The initial value in this example is $x_0 = 1$.

The iterative process is repeated and the value of x_n tends to the fixed point of the recurrence relation as n $\Rightarrow \infty$. This is also the solution to the equation x = -0.5x + 6 which is x = 4

Notice that the fixed point is where the lines y = x and y = -0.5x = 6 intersect.

Staircase and Cobweb diagrams

15 min

Q25: Draw Staircase or Cobweb diagrams for the following recurrence relations. For each diagram write down the fixed point and determine whether the recurrence relation leads to this fixed point or not.

a) $x_{n+1} = 0.2x_n + 4$, $x_0 = 0$

b) $x_{n+1} = -0.5x_n + 12$, $x_0 = 1$

c) $x_{n+1} = 1.5x_n + 6$, $x_0 = 1$

d) $x_{n+1} = -2x_n + 12$, $x_0 = 2$

3.2.3 The iterative scheme $x_{n+1} = g(x_n)$

If we require to obtain solutions for the equation $x^2 + 3x - 2 = 0$ we can use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and so obtain the solutions

$$x = \frac{1}{2}\left(-3 \pm \sqrt{17}\right)$$

However, suppose that we wish to find the solutions of equations such as $x^3 + 2x^2 + 10x - 20 = 0$, $xe^x = 1$, or $x = 3 \sin x$ then we have no convenient formula and so we need to use another method. One technique for solving such equations is to use an iterative method. We rewrite the original equation in the form $x = g(x)$ and then we set up the corresponding recurrence relation $x_{n+1} = g(x_n)$. Taking x_0 as an initial approximation to the solution we are then able to calculate $x_1, x_2, x_3, \ldots$ and hopefully this will lead to a fixed point and therefore a solution to the original equation. However you should be wary from our previous work on first order linear recurrence relations that this may not always be the case. The following activity should highlight some of the difficulties you may encounter.

Testing Convergence

25 min

Learning Objective

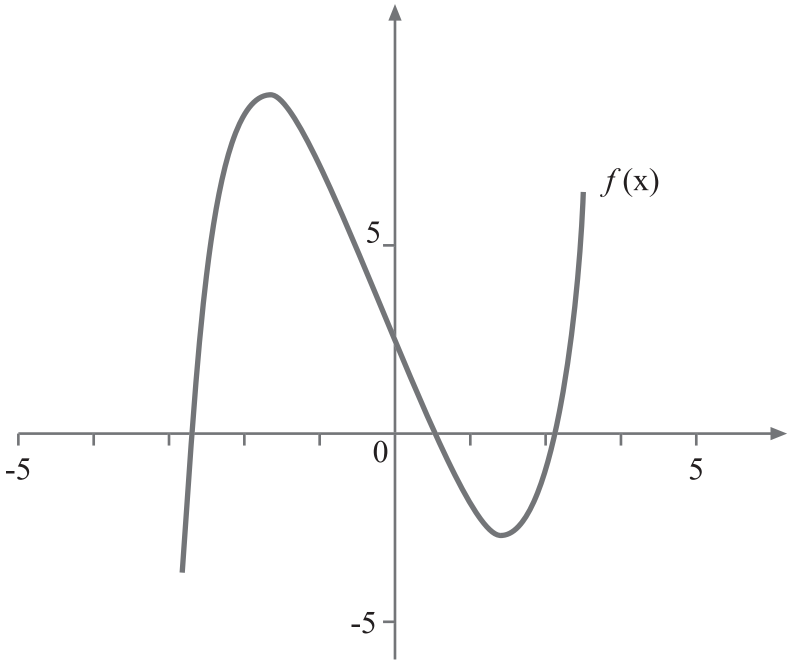
Evaluate different iterative processes.

Consider the cubic equation

$f(x) = x^3 - 6x + 3$

Using a graphic calculator it is possible to make a sketch of this graph and therefore obtain an estimate for the three roots of the equation.

Your graph should look something like this



You can see that a reasonable estimate for the three roots is

α = -2.7, β = 0.5, δ = 2.1

Now to obtain a more accurate value for these roots we can attempt to apply an iterative process. To achieve this we rearrange $x^3 - 6x + 3 = 0$ as $x = g(x)$. This can be done in many ways.

Check that the following are all possible rearrangements.

(1) $x = \dfrac{x^3 + 3}{6}$

(2) $x = \dfrac{3}{6 - x^2}$

(3) $x = x^3 - 5x + 3$

(4) $x = \dfrac{2x^3 - 3}{3x^2 - 6}$

You can probably think of some more yourself.

Using one of these rearrangements we can now set up an iterative process to try to obtain the roots of the original equation to a greater degree of accuracy.

Rearrangement (1) will give us the iterative scheme

$x_{n+1} = \frac{x_n^3 + 3}{6}$

We can now check to see how effective this is for finding the roots.

Let $x_0 = 0.5$ then we obtain the following results (correct to 6 decimal places).

x_0	0.5
x_1	0.520833
x_2	0.523548
x_3	0.523918
x_4	0.523968
x_5	0.523975
x_6	0.523976
x_7	0.523976
...	...
x_n	0.523976

Therefore it seems that this iterative process has converged to the root near 0.5 and, correct to 6 decimal places, this root is 0.523976

(A graphic calculator can do these calculations very quickly. For example, we can achieve the result using the following steps

- key in x_0 = 0.5 and press enter.
- now key in (ANS^3+3)/6
- Pressing enter repeatedly will give x_1, x_2, x_3, ...)

However, look what happens if we try to find the root near -2.7

x_0	- 2.7
x_1	- 2.7805
x_2	- 3.082758
x_3	- 4.382778
x_4	- 13.531274
x_5	- 412.418921
x_6	- 11691345.03
...	...
x_n	$\Rightarrow \infty$

For x_0 = -2.7 the sequence diverges and we are unable to find a more accurate value for the root near -2.7

Q26: Using the same iteration formula as above find out what happens when you take x_0 = 2.1

Having considered different iterative schemes with different values for x_0 it should now be obvious to you that not every iterative scheme will converge to the required solution. Convergence will depend on the iterative scheme and on the value of x_0 used.

Q27: There is an interactive version of this question on the course web site.

Copy and complete the following table to show how each iteration formula behaves near the given values of x_0

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	$x_0 = -2.7$	$x_0 = 0.5$	$x_0 = 2.1$
$x_{n+1} = \frac{x_n^3 + 3}{6}$	Diverges	Converges to 0.523976	
$x_{n+1} = \frac{3}{6 - x_n^2}$			
$x_{n+1} = x^3 - 5x_n + 3$			
$x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 6}$			

Q28: Which of the previous iterative schemes gave all three solutions?

25 min

Exercise 5

An on-line assessment is available at this point, which you might find helpful.

Q29: The equation $xe^x = 1$ has one solution near $x = 0.5$

Find an iterative scheme that will find this solution, and write down your answer to 6 decimal places.

Q30: The equation $\sin x + x - 2 = 0$ has one solution near $x = 1$

Find an iterative scheme that will find this solution and write down your answer to 6 decimal places.

Q31: The equation $x^3 + x - {}^1/_2$ has one solution near $x = 0$

$x = {}^1/_2 - x^3$ and $x = \left(\frac{1}{2} - x\right)^{1/3}$ are both rearrangements of the above equation.

What results do you obtain when you apply iterative schemes to these rearrangements? If possible write down the value of the solution to 6 decimal places.

Q32: The equation $e^x - x = 3$ has two solutions, one near -3 and the other near 1.5

Show that $x = e^x - 3$ and $x = \ln(x + 3)$ are both rearrangements of this equation.

What results do you obtain when you apply iterative schemes to these rearrangements? Give any solutions to 6 decimal places.

3.2.4 Locating an initial approximation

Learning Objective

Use a graphical technique to locate the approximate solution x_0

In practice we may often obtain an initial approximation to roots of the equation $f(x) = 0$ by using a graphic calculator to graph $y = f(x)$ and then reading the x-coordinates of the point(s) where the curve cuts the x-axis.

It is useful to have another method which will allow us to approximate x_0 without relying on a graphic calculator. The following example shows how this is possible.

Example Find an approximate solution to the equation $x^3 - 2x + 3 = 0$

Answer

We can rewrite $x^3 - 2x + 3 = 0$ as $x^3 = 2x - 3$

Now if we graph $y = x^3$ and $y = 2x - 3$ on the same diagram then the intersection of these two lines will give an approximate solution for $x^3 - 2x + 3 = 0$

(Since $y = x^3$ and $y = 2x - 3$ are familiar graphs, this is probably a better method than trying to graph $y = x^3 - 2x + 3$).

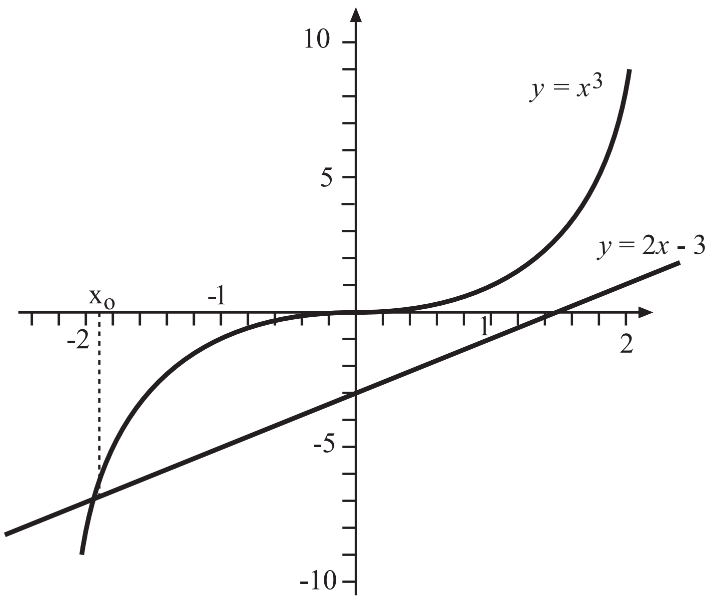
You should try to draw these graphs as accurately as you can so that you can achieve a good estimate for x_0 (2mm graph paper is recommended)

A table of values will help and then the graph can be plotted more accurately.

x	- 2	- 1.5	- 1	- 0.5	0	0.5	1	1.5	2
$y = x^3$ (2 d.p.)	- 8	- 3.38	- 1	- 0.13	0	0.13	1	3.38	8

x	- 2	- 1.5	- 1	- 0.5	0	0.5	1	1.5	2
y = 2x - 3	- 7	- 6	- 5	- 4	- 3	- 2	- 1	0	1

Notice that the scales on the axes are different. We need to choose a large scale for the x -axis so that we can achieve a good estimate for x_0



From the graph it would seem that there is only one solution for the equation and a first approximation to this solution is $x_0 = -1.9$

To find the solution to a greater degree of accuracy we then use a suitable iterative scheme with $x_0 = -1.9$

You can check that $x = (2x - 3)^{1/3}$ is a rearrangement of $x^3 - 2x + 3 = 0$

The iterative scheme $x_{n+1} = (2x - 3)^{1/3}$ with $x_0 = -1.9$ then gives us the root more accurately. To 6 decimal places the root is -1.893289

Exercise 6

20 min

An on-line assessment is available at this point, which you might find helpful.

Q33: a) Use a graphical method to find an approximate solution for the equation

$e^x - x - 4 = 0$ between 1 and 2.

b) The equation $e^x - x - 4 = 0$ can be rewritten as $x = \ln (x + 4)$. Use your value for x_0 and the iterative scheme $x_{n+1} = \ln (x_n + 4)$ to find the solution of the equation to 6 decimal places.

Q34: a) The equation $x^3 - x + 4 = 0$ has one real solution. Use a graphical method to obtain an approximation to this solution.

b) Find an iterative scheme that will give this solution more accurately and write down the value of the solution to 6 decimal places.

Q35:

a) Show graphically that the equation $\ln x + x - 3 = 0$ has only one real solution and find an approximate value for this solution.

b) Find an iterative scheme that will give this solution more accurately and write down the value of the solution to 6 decimal places.

Q36:

a) Show graphically that the equation $x^4 - 5x + 1 = 0$ has two real solutions for

$0 <\leq x \leq 2$ and write down an approximate value for these solutions.

b) Find iterative schemes that will give you these solutions to 6 decimal places.

3.2.5 Conditions for convergence

Learning Objective

Determine when an iterative process will converge to a solution

From the work that we have done up to now it should be obvious that the success of an iterative scheme in the form $x_{n+1} = g(x_n)$ depends on:

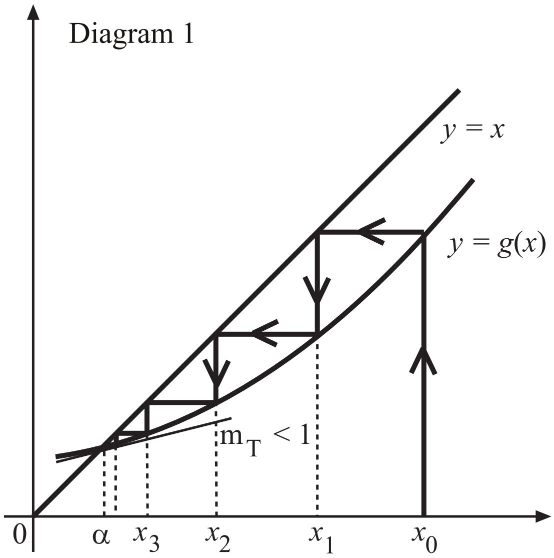
i the starting value used

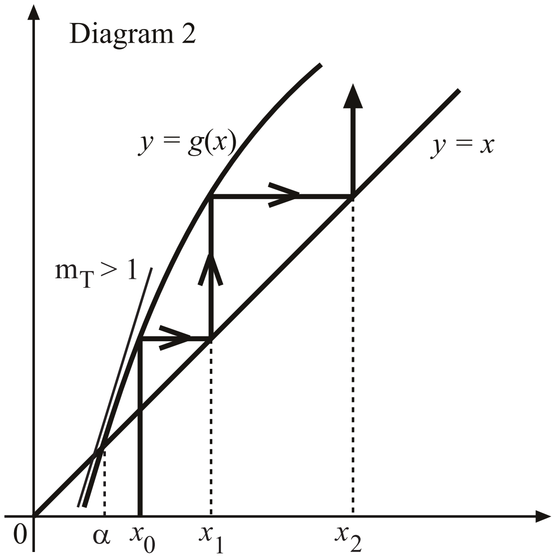
ii and the particular iterative formula used.

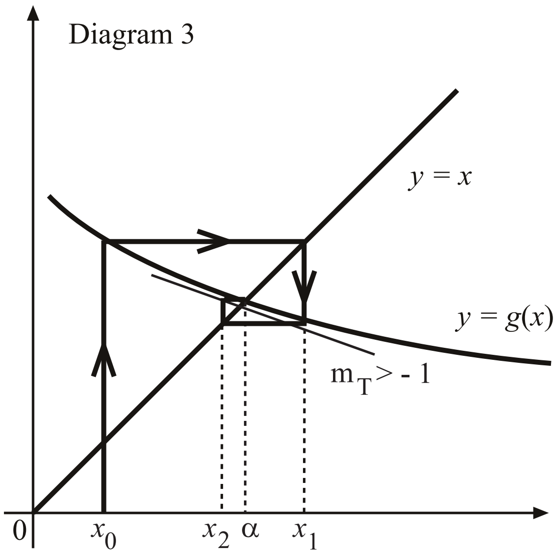
When we were dealing with a linear first order recurrence relation we were able to determine the condition that $-1 < a < 1$ for the recurrence relation $u_{n+1} = au_n + b$ to converge to a limit.

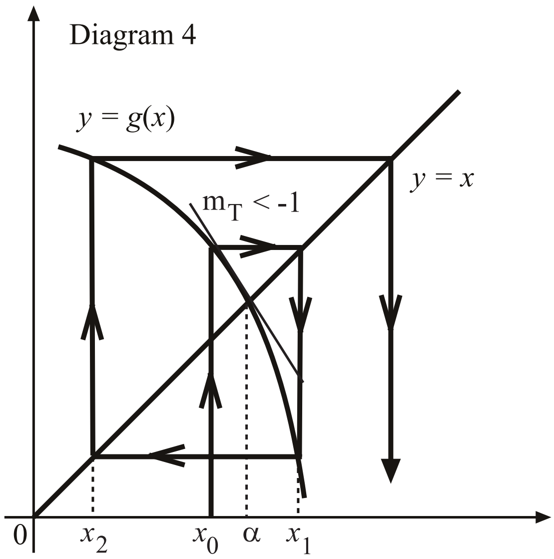
It would be useful now if we were able to determine for the iterative process $x_{n+1} = g(x_n)$ whether or not a particular formula and starting value will converge to the appropriate solution. In a similar way for recurrence relations we can represent $x_{n+1} = g(x_n)$ geometrically in either a staircase or cobweb diagram by the lines $y = x$ and y = g (x)

Study the following diagrams and try to arrive at some conclusions. The solution of the equation x = g (x) is where the two lines intersect at the fixed point α. Notice that the gradient of the curve y = g (x_n) at the fixed point is indicated by m_T









In diagrams 1 and 3 the iterative process converges towards the fixed point α. In diagrams 2 and 4 the iterative process diverges away from the fixed point α

Hopefully you will have spotted that the iterative process converges towards the fixed point α when $-1 < m_T < 1$ ie. $|\ m_T\ | < 1$ near the fixed point.

Therefore the iterative process $x_{n+1} = g\ (x_n)$ converges towards the fixed point α when $|\ g'\ (x)\ | < 1$ in an interval close to α

Example $x = \frac{2x^3 + 2}{5}$, $x = \left(\frac{5x - 2}{2}\right)^{\frac{1}{3}}$, and $x = \frac{4x^3 - 2}{6x^2 - 5}$

are all rearrangements of the equation $2x^3 - 5x + 2 = 0$

These rearrangements give the iterative process $x_{n+1} = g\ (x_n)$

The equation has three solutions. One of these solutions is near 0.5

By considering the value of $|\ g\ '\ (x)\ |$ at 0.5 for the iterative processes determine which of them will converge to the solution near 0.5

Answer

1. For the iterative scheme

   $x_{n+1} = \frac{2x_n^3 + 2}{5}$

   we have

   $g\ (x) = \frac{2x^3 + 2}{5}$

   $g\ '\ (x) = {}^6/_5 x^2$ and $g\ '\ (0.5) = 0.3$

   Since $|\ g\ '\ (0.5)\ | < 1$ we can expect this iteration to converge to the solution near 0.5

2. For the iterative scheme

   $x_{n+1} = \left(\frac{5x_n - 2}{2}\right)^{\frac{1}{3}}$

   we have

   $g\ (x) = \left(\frac{5x - 2}{2}\right)^{\frac{1}{3}}$

   $g'\ (x) = \frac{5}{6}\left(\frac{5x - 2}{2}\right)^{-\frac{2}{3}}$ and $g\ '\ (0.5) \approx 2.1$

   Since $|\ g\ '\ (0.5)\ | > 1$ we should not expect this iterative scheme to converge to the solution near 0.5

3. For the iterative scheme

   $x_{n+1} = \frac{4x_n^3 - 2}{6x_n^2 - 5}$

   we have

   $g\ (x) = \frac{4x^3 - 2}{6x^2 - 5}$

   $g'\ (x) = \frac{12x^2(6x^2 - 5) - 12x(4x^3 - 2)}{(6x^2 - 5)^2}$

   and $g\ |\ (0.5) = -0.1$ approx

   Since $|\ g\ '\ (0.5)\ | < 1$ we can expect this iteration to converge to the solution near 0.5

Using either of the iterative methods in 1 or 3 we are able to calculate the root as 0.432320 (to 6 decimal places).

You should also note that unlike a linear first order recurrence relation it will make a difference to the iterative scheme $x_{n+1} = g\ (x_n)$ which value we choose for x_0. Since $g\ (x)$ represents the equation of a curve its gradient will vary, so given that the iterative process converges and $|\ g\ '\ (\alpha)\ | < 1$, x_0 should be chosen as close to the root α as possible to ensure that $|\ g\ '\ (x_0)\ | < 1$ also.

Now try the questions in the following exercise.

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Exercise 7

An on-line assessment is available at this point, which you might find helpful.

20 min

Q37:

The equation $4x + 5 - x^3 = 0$ has one solution near $x_0 = 2$

a) Show that the formula $x_{n+1} = (4x_n + 5)^{1/3}$ with $x_0 = 2$ should give a convergent process for this solution.

b) Does the formula $x_{n+1} = \frac{1}{4}\left(x_n^3 - 5\right)$ with $x_0 = 2$ converge?

Q38:

The equation $x^3 - 4x - 1 = 0$ has three solutions, near -1.9, -0.3 and 2.1

$x_{n+1} = \frac{1}{x_n^2 - 4}$ is an iterative scheme for this equation in the form $x_{n+1} = g(x_n)$

a) Calculate $g'(x_0)$ for the initial values $x_0 = -1.9, -0.3$ and 2.1 and therefore determine for which of these values you would expect the iterative formula to converge to the required solution.

b) Calculate the value of the solution or solutions that this iterative scheme gives you (to 6 decimal places).

Q39:

a) The following are all rearrangements of the equation $2x^3 - 5x + 2 = 0$

(A) $x = \frac{2x^3 + 2}{5}$, (B) $x = \left(\frac{5x - 2}{2}\right)^{\frac{1}{3}}$, and (C) $x = \frac{4x^3 - 2}{6x^2 - 5}$

These rearrangements give the iterative process $x_{n+1} = g(x_n)$

The equation has three solutions. One of these solutions is near -1.8

By considering the value of $|g'(x)|$ at -1.8 for the above iterative processes determine which of them will converge to the solution near -1.8

b) Calculate the value of this solution to 6 decimal places.

3.2.6 Order of convergence

Learning Objective

Determine the order of convergence for an iterative process

We now have a way for deciding whether a particular iterative process will converge or not to a particular solution. You may also have noticed that some iterative processes converge more quickly than others. This depends on the **order of convergence**. The following example should explain this.

Example The equation $x^2 - 7 = 0$ has two solutions, one of which is near 2.6 and the other near -2.6. Notice that solving this equation will give the exact roots $\alpha = \pm\sqrt{7}$

Check that the following are both rearrangements of this equation

$x = \frac{x + 7}{x + 1}$ and $x = \frac{1}{2}\left(x + \frac{7}{x}\right)$

For $x_{n+1} = \dfrac{x_n + 7}{x_n + 1}$, $g(x) = \dfrac{x + 7}{x + 1}$

$$\text{and } g'(x) = \frac{-6}{(x+1)^2}$$

$$g'(2.6) = -0.463 \text{ approx}$$

For $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{7}{x_n}\right)$, $g(x) = \dfrac{1}{2}\left(x + \dfrac{7}{x}\right)$

$$\text{and } g'(x) = \frac{1}{2}\left(1 - \frac{7}{x^2}\right)$$

$$g'(2.6) = -0.018 \text{ approx}$$

Since $|\ g'(2.6)\ | < 1$ for both of these iterative processes they converge to the solution near 2.6 as follows.

The iteration $x_{n+1} = \dfrac{x_n + 7}{x_n + 1}$

gives the following iterates

x_0	2.6
x_1	2.666667
x_2	2.636364
x_3	2.650000
x_4	2.643836
x_5	2.646617
x_6	2.645361
x_7	2.645928
x_8	2.645672
x_9	2.645787
x_{10}	2.645735
x_{11}	2.645759
x_{12}	2.645748
x_{13}	2.645753
x_{14}	2.645751
x_{15}	2.645752
x_{16}	2.645751
x_{17}	2.645751

The iteration $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{7}{x_n}\right)$

gives the following iterates

x_0	2.6
x_1	2.646154
x_2	2.645751
x_0	2.645751

Both iterations give the solution 2.645751 (to 6 decimal places). However, it is very obvious that the second iterative process converges far more quickly.

The value of $|\ g'(x)\ |$ gives some indication of the speed of convergence for an iterative process. Notice that for the second iteration $|\ g'(2.6)\ |$ was much closer to zero. This indicates that it should converge more quickly.

In this case we can actually calculate $g'(\alpha)$ since $\alpha = \sqrt{7}$

For $g'(x) = \dfrac{-6}{(x+1)^2}$

$$g'(\alpha) = \frac{-6}{(\sqrt{7}+1)^2} = -0.451 \text{ approx}$$

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For $g'(x) = \frac{1}{2}\left(1 - \frac{7}{x^2}\right)$

$$g'(\alpha) = \frac{1}{2}\left(1 - \frac{7}{\sqrt{7}^2}\right) = 0$$

The previous two iterative processes are examples of first order and second order convergence.

$x_{n+1} = \frac{x_n + 7}{x_n + 1}$ is an iteration with first order convergence

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$ is an iteration with second order convergence.

order of convergence

For an iterative process in the form $x_{n+1} = g(x_n)$

The **order of convergence** is first order when $|g'(x)| < 1$ but $|g'(\alpha)| \neq 0$

The order of convergence is second order when $|g'(x)| < 1$ and $|g'(\alpha)| = 0$ but $|g''(\alpha)| \neq 0$

Similar statements can also be made about higher orders of convergence.

Now try the following questions.

Exercise 8

An on-line assessment is available at this point, which you might find helpful.

15 min

Q40: Show that the following iterative processes have first order convergence i.e. show that $|g'(x)| < 1$ and that $g'(\alpha) \neq 0$

a) $x_{n+1} = \frac{2}{x_n + 4}$ for the solution of $x^2 + 4x - 2 = 0$ near $x = 0$

b) $x_{n+1} = 1 - \sin x_n$ for the solution of $\sin x + x - 1 = 0$ between 0 and 1 radians

Q41: The iterative process $x_{n+1} = \frac{1}{2}x_n\left(3 - 5x_n^2\right)$ can be used to determine $\frac{1}{\sqrt{5}}$. Show that this process is second order and apply it to obtain $\frac{1}{\sqrt{5}}$ to 6 decimal places. (You can check this on your calculator.)

3.3 Summary

Learning Objective

Recall the main learning points from this topic

1. A power series is an expression of the form

   $$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

   where $a_0, a_1, a_2, ..., a_n, ...$ are constants and x is a variable.

2. The Maclaurin series generated by the function f (x) is

   $$\sum_{r=0}^{\infty} f^{(r)}(0)\frac{x^r}{r!} = f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + ... + f^{(n)}(0)\frac{x^n}{n!} + ...$$

3. For the linear recurrence relation $u_{n+1} = au_n + b$, u_n tends to a limit, L, when $-1 < a < 1$

   L is given by the formula $L = \frac{b}{1-a}$

4. The iterative process $x_{n+1} = g(x_n)$) will converge to the fixed point α when $|g'(x)| < 1$ for x close to α

5. The order of convergence for $x_{n+1} = g(x_n)$ is first order when $|g'(x)| < 1$ but $|g'(\alpha)| \neq 0$

   The order of convergence for $x_{n+1} = g(x_n)$ is second order when $|g'(x)| < 1$ and $|g'(\alpha)| = 0$ but $|g''(\alpha)| \neq 0$

3.4 Extended information

There are links on-line to a variety of web sites related to this topic.

Colin Maclaurin (1698 - 1746)

Colin Maclaurin was born in Kilmodan, Scotland. He was the youngest of three brothers but never knew his father who died when he was only six weeks old. His mother also died, when Colin was nine years old, and so he was brought up by his uncle who was a minister at Kifinnnan on Loch Fyne.

At the age of 19, in August 1717, Maclaurin was appointed professor of mathematics at Marischal College, Aberdeen University. Maclaurin at this time was a great supporter of Sir Isaac Newton and is reported to have travelled to London to meet him. He was elected a Fellow of the Royal Society during one of these visits.

On 3 November 1725 Maclaurin was appointed to Edinburgh University where he spent the rest of his career. He married Anne Stewart the daughter of the Solicitor General for Scotland and had seven children. His teaching at Edinburgh came in for considerable praise and he is said to have been keen to aid the understanding of his students. If they had difficulty with a concept then he was likely to try another method of explanation in order to give them a clearer understanding.

Maclaurin is best remembered for his publication the *Treatise of fluxions* where he demonstrates the special case of the Taylor series which is now named after him.

The Taylor series generated by the function f (x) at x = a is

$$f(a) + f^{(1)}(a)(x - a) + f^{(2)}(a)\frac{(x - a)^2}{2!} + \dots + f^{(n)}(a)\frac{(x - a)^n}{n!} + \dots$$

Notice that Maclaurin series are Taylor series with a = 0. The Maclaurin series was not an idea discovered independently of the Taylor series and indeed Maclaurin makes acknowledgement of Taylor's influence.

Maclaurin's other interests include the annual eclipse of the sun, the structure of bees' honeycombs and actuarial studies.

'He laid sound actuarial foundations for the insurance society that has ever since helped the widows and children of Scottish ministers and professors.'

Maclaurin also became involved in the defence of Edinburgh during the Jacobite rebellion of 1745. However, when the city fell to the Jacobites he fled to England but returned when the Jacobites marched further south. Much weakened by a fall from his horse in combination with his exertions defending Edinburgh and a difficult journey through winter weather to return to his home city, he became very ill in December 1745. He died the next year and was buried in Greyfriars Churchyard, where his grave can still be seen.

Calculating π

The series for $\tan^{-1}x$ was first discovered by James Gregory in 1671.

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

When x = 1 in the above then we obtain Leibniz's formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\frac{(-1)^{n-1}}{2n - 1} + \dots$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\frac{(-1)^{n-1}}{2n - 1} + \dots\right)$$

This series converges very slowly and so is not used to approximate π to many decimal places. The series for $\tan^{-1}x$ converges more quickly when x is near zero. Therefore if you want to use the series for $\tan^{-1}x$ to calculate π then you could consider various trigonometric identities.

For example we could use the following trigonometrical identity with

$\alpha = \tan^{-1}\frac{1}{2}$ and $\beta = \tan^{-1}\frac{1}{3}$

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{(1/2) + (1/3)}{1 - (1/6)} \\ &= 1 \\ &= \tan\frac{\pi}{4} \end{aligned}$$

Therefore

$$\frac{\pi}{4} = \alpha + \beta = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

$$\pi = 4\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$$

Now we can use the expansion for $\tan^{-1}x$ with $x = {}^1/_2$ and $x = {}^1/_3$

Since these values for x are nearer to zero the above method will give accurate results for π more quickly than Leibniz's formula.

3.5 Review exercise

Review exercise in further sequences and series

15 min

An on-line assessment is available at this point, which you might find useful.

Q42: Find the first three terms of the Maclaurin series for f (x) = exp (3x)

Q43: The equation $x^3 - 3x + 2 = 0$ can be rewritten as $x = (3x - 3)^{1/3}$. This equation has a solution which lies in the interval $-3 < x < -2$

By using the simple iterative scheme $x_{n+1} = (3x_n - 3)^{1/3}$ with $x_0 = -2$, find an approximation to this solution. Give your answer correct to three decimal places.

3.6 Advanced review exercise

Advanced review exercise in further sequences and series

30 min

An on-line assessment is available at this point, which you might find helpful.

Q44: Use Maclaurin's theorem to write down the expansion, as far as the term in x^4, of $\sqrt{1 + 2x}$, $|x| < \frac{1}{2}$

Q45: Use Maclaurin's theorem to write down the expansion, as far as the term in x^3, of $(3 - x)^{-2}$, $|\ x\ | < 1$

Q46: The equation $xe^x = 2$ can be rewritten as $x = 2e^{-x}$ and has one solution near x = 1

a) Show that the iterative scheme $x_{n+1} = 2e^{-x_n}$ will converge to give the solution near x = 1 and calculate the value of this solution to 3 decimal places.

b) Determine the order of convergence for this iterative scheme.

Q47: The solution of the equation $x^3 - 6 = 0$ will give the cube root of 6

a) Show that $x = \frac{2x^3 + 6}{3x^2}$ is a rearrangement of this equation.

Use the iterative scheme, which comes from this rearrangement with $x_0 = 2$, to write down the values you get for x_0, x_1, x_2, x_3, x_4, ... Find the cube root of 6 to 3 decimal places.

b) Determine the order of convergence for this iterative scheme.

3.7 Set review exercise

Set review exercise in further sequences and series

10 min

An on-line assessment is available at this point, which you will need to attempt to have these questions marked. These questions on the web site are not randomised and may be posed in a different manner, but you should have the required answers in your working.

Q48: Find the first three terms of the Maclaurin series for f (x) = sin 3x

Q49: The equation $x^3 + 2x - 4 = 0$ can be rewritten as $x = (4 - 2x)^{1/3}$. This equation has a solution which lies in the interval $1 < x < 2$

By using the simple iterative scheme $x_{n+1} = (4 - 2x_n)^{1/3}$ with $x_0 = 1$, find an approximation to this solution. Give your answer correct to three decimal places.

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Topic 4

Further Ordinary Differential Equations

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Learning Objectives

Solve further ordinary differential equations.

Minimum Performance Criteria:

- *Solve first order linear differential equations using the integrating factor method.*

Prerequisites

Before attempting this unit you should be familiar with:

- The rules for differentiating and integrating functions and in particular the product rule and chain rule.
- Properties of the exponential and logarithmic functions and in particular $a \ln x = \ln x^a$ and $e^{\ln x} = x$
- Solving quadratic equations and using the quadratic formula.
- The complex number i and its use when evaluating the square root of a negative number.

Try the following exercise to test your skills.

Some revision may be necessary if you find this difficult.

15 min

Revision exercise

These questions are designed to practice skills that you should already have. If you have difficulty you could consult your tutor or a classmate.

Q1: For $f(x) = x^2 \sin x$ calculate $f'(x)$

Q2: For $y = e^{4x}$ calculate dy/dx

Q3: Integrate $\int \frac{3}{x} dx$

Q4: Simplify $e^{3 \ln x}$

Q5: Rewrite $\sqrt{(-49)}$ in terms of i

Q6: Find the complex roots of the equation $x^2 + 4x + 13 = 0$

4.1 First order linear differential equations

4.1.1 Introduction

Learning Objective

Recognise a first order linear differential equation

first order linear differential equation

A **first order linear differential equation** is an equation that can be expressed in the standard form

$dy/dx + P(x)y = f(x)$

In general, such equations contain only simple terms in y such as y and dy/dx and not more complicated nonlinear terms such as y^2, $y\, dy/dx$, e^y, sin y etc

The examples that follow are first order linear differential equations.

Examples

1. $^{dy}/_{dx} + 3xy = x$

($P(x) = 3x$, $f(x) = x$)

2. $^{dy}/_{dx} - y = ex$

($P(x) = -1$, $f(x) = e^x$)

3. $^{dy}/_{dx} + y \sin t = t^2$

($P(t) = \sin t$, $f(t) = t^2$)

Some algebraic manipulation may be necessary in order to express an equation in this form.

Examples

1. $x^{dy}/_{dx} - 3x^2 = y$ can be expressed as $^{dy}/_{dx} - {}^{y}/_{x} = 3x$

2. $^{dy}/_{dx} = x^2 + xy - 3$ can be expressed as $^{dy}/_{dx} - xy = x^2 - 3$

4.1.2 Solving first order linear equations

Learning Objective

Solve first order linear differential equations using the integrating factor method

The first order linear differential equation $^{dy}/_{dx} + P(x)y = f(x)$ can be solved by multiplying both sides of the equation by a suitable function called the **integrating factor**, $I(x)$

integrating factor

For the first order linear differential equation $^{dy}/_{dx} + P(x)y = f(x)$, the **integrating factor** is given by $I(x) = e^{\int P(x)\,dx}$

Note that

$$\frac{d}{dx}\left(I(x)\right) = \frac{d}{dx}\left(e^{\int P(x)\,dx}\right) = P(x)\,e^{\int P(x)dx} = P(x)\,I(x)$$

Here we have used

- the chain rule

  $\frac{d}{dx}\left(e^{u(x)}\right) = u'(x)e^{u(x)}$ $\left(\text{where } u(x) = \int P(x)dx\right)$

- and the fact that

  $\frac{d}{dx}\left(\int P(x)dx\right) = P(x)$

Using the the product rule and then $\frac{d}{dx}(I(x)) = \frac{d}{dx}\left(e^{\int P(x)\,dx}\right) = P(x)\,e^{\int P(x)dx} = P(x)\,I(x)$ we obtain

$$\begin{aligned}\frac{d}{dx}[I(x)y] &= \frac{d\,(I(x))}{dx}y + I(x)\frac{dy}{dx}\\ &= P(x)\,I(x)y + I(x)\frac{dy}{dx}\end{aligned}$$

We can now see why multiplying both sides of ${}^{dy}/_{dx} + P(x)y = f(x)$ by $I(x)$ enables us to solve the equation.

If $\frac{dy}{dx} + P(x)y = f(x)$

then $I(x)\frac{dy}{dx} + I(x)\,P(x)y = I(x)\,f(x)$

and so $\frac{d}{dx}[I(x)y] = I(x)\,f(x)$

Integrating both sides we obtain

$I(x)y = \int I(x)\,f(x)dx$

which can now be solved for y

Hence a first order differential equation can be solved using the following strategy.

Strategy for solving first order linear differential equations

1. Write the equation in standard linear form

   $\frac{dy}{dx} + P(x)y = f(x)$

   and thus identify $P(x)$ and $f(x)$
2. Calculate the integrating factor $I(x) = e^{\int P(x)dx}$
3. Mutiply the equation in standard linear form by $I(x)$ and obtain the equation

   $\frac{d}{dx}[I(x)y] = I(x)\,f(x)$
4. Integrate both sides to give

   $I(x)y = \int I(x)\,f(x)dx$
5. Rearrange to solve for y.

general solution

The **general solution** of a differential equation contains one or more arbitrary constants and gives infinitely many solutions that all satisfy the differential equation.

Example Find the **general solution** of

${}^{dy}/_{dx} + 2y = e^{3x}$

Answer

1. Note that the equation is given in standard linear form so we can easily identify $P(x) = 2$ and $f(x) = e^{3x}$
2. We can now calculate the integrating factor
   $I(x) = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$
   Note that we do not use a constant of integration in finding the integrating factor.
3. Now use the fact that the equation can be written as
   $\frac{d}{dx}[I(x)y] = I(x)f(x)$
   With $I(x) = e^{2x}$ and $f(x) = e^{3x}$ the equation becomes
   $$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}e^{3x}$$
   $$\frac{d}{dx}\left(e^{2x}y\right) = e^{5x}$$
4. Integrating both sides of this last equation gives
   $$\int \frac{d}{dx}\left(e^{2x}y\right)dx = \int e^{5x}dx$$
   $$e^{2x}y = \frac{1}{5}e^{5x} + C$$
5. Rearranging to solve for y we obtain the **general solution**
   $y = {}^{1}/_{5}e^{3x} + Ce^{-2x}$

Note that in order to apply this method it is first necessary to ensure that the equation is expressed in standard linear form.

${}^{dy}/_{dx} + P(x)y = f(x)$

See the following example.

Example Find the general solution of

$x\,{}^{dy}/_{dx} + y = \sin x$

Answer

We can rewrite the equation in standard linear form as ${}^{dy}/_{dx} = {}^{y}/_{x} = {}^{\sin x}/_{x}$

Thus $P(x) = {}^{1}/_{x}$

Hence the integrating factor is $I(x) = e^{\int 1/x dx} = e^{\ln x} = x$

Multiplying both sides of ${}^{dy}/_{dx} + {}^{y}/_{x} = {}^{\sin x}/_{x}$ by $I(x)$ we obtain

$$\frac{d}{dx}[I(x)y] = I(x)\frac{\sin x}{x}$$
$$\text{i.e. } \frac{d}{dx}(xy) = \frac{x\sin x}{x} = \sin x$$

Integrating we obtain

$xy = \int \sin x dx = -\cos x + C$

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Rearranging we obtain the **general solution**

$y = -\,{}^{\cos x}/_{x} + {}^{C}/_{x}$

We saw in Unit 2, Further Integration (7.4) that if we are given an **initial condition** for a first order differential equation then it should be possible to find a **particular solution** of the differential equation satisfying the initial condition.

initial condition

For a differential equation an **initial condition** is an additional condition which must be satisfied by the solution. This could be a coordinate on a curve, a velocity at *t* = 0, the amount of money in a bank account on 1st January 2000, etc.

particular solution

The **particular solution** of a differential equation is a solution which is often obtained from the general solution when values are assigned to the arbitrary constants.

The following example illustrates this point.

Example Solve the initial value problem

$$\begin{cases} x\dfrac{dy}{dx} + 3y = 5x^2 \\ y\,(1) = 0 \end{cases}$$

Answer

In standard linear form the equation is

${}^{dy}/_{dx} + {}^{3y}/_{x} = 5x$

Thus $P\,(x) = {}^{3}/_{x}$

Therefore the integrating factor is $I\,(x) = e^{\int 3/x dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

Multiplying both sides of ${}^{dy}/_{dx} + {}^{3y}/_{x} = 5x$ by $I\,(x)$ we obtain

$$\begin{aligned} \frac{d}{dx}\left[I\,(x)y\right] &= I\,(x)5x \\ \frac{d}{dx}\left(x^3y\right) &= x^3.5x = 5x^4 \end{aligned}$$

Integrating we obtain $x^3y = \int 5x^4dx = x^5 + C$

Now we are given that $y\,(1) = 0$ and so we can find C

Substituting $x = 1$ and $y = 0$ into $x^3y = x^5 + C$ gives $0 = 1 + C$ so $C = -1$

Therefore $x^3y = x^{5+C}$ becomes $x^3y = x^5 - 1$ and rearranging to solve for y gives

$y = x^2 - x^{-3}$

Now try the questions in Exercise 1.

40 min

Exercise 1

An on-line assessment is available at this point, which you might find helpful.

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Use the strategy given previously to find a general solution for the following differential equations.

Q7: $^{dy}/_{dx} - 2y = e^{3x}$

Q8: $^{dy}/_{dx} + {}^{y}/_{x} = 4x^2$

Q9: $2^{dy}/_{dx} + y = e^{x/2}$

Q10: $x^{dy}/_{dx} + 2y = {}^{\cos x}/_{x}$

Q11: $\frac{dy}{dx} = \frac{xy + 4}{x^2}$

Q12: $x\frac{dy}{dx} + y = x\sin x$

Solve the following initial value problems.

Q13:

$$\begin{cases} \dfrac{dy}{dx} + \dfrac{2y}{x} = 6 \\ y(1) = 1 \end{cases}$$

Q14: (Note that this linear equation is also a separable equation and so there is a choice of methods for solving it.)

$$\begin{cases} \dfrac{dy}{dx} + 5y = 2 \\ y\left(\dfrac{1}{5}\right) = \dfrac{3}{5} \end{cases}$$

Q15:

$$\begin{cases} \dfrac{dy}{dx} - \cos(x)y = 2x\, e^{\sin(x)} \\ y(\pi) = 0 \end{cases}$$

Q16:

$$\begin{cases} x^2\dfrac{dy}{dx} + y = x^2 e^{1/x} \\ y(1) = 0 \end{cases}$$

Q17:

$$\begin{cases} \dfrac{dy}{dx} + 3x^2 y = 6x^2 \\ y(0) = 1 \end{cases}$$

Extra Help: Differential Equations - linear 1

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

Extra Help: Differential Equations - linear 2

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

4.1.3 Applications

Learning Objective

Solve differential equations in context

As we have already seen in Unit 2, Further Integration, differential equations have important applications in engineering and science. In this section we look at some further examples that involve first order linear equations.

Examples

1. Bacterial Growth

Bacteria grow in a certain culture at a rate proportional to the amount of bacteria present. Given that there are 100 bacteria present initially, find an equation for bacterial growth and by solving an appropriate differential equation find the number of bacteria present at any subsequent time.

Answer

We have already seen how to deal with this type of problem when we used the method of separating variables to solve first order differential equations. However, we can use an integrating factor as an alternative method.

Since bacteria grow at a rate proportional to the amount of bacteria present, the differential equation describing bacterial growth is

$^{dB}/_{dt} = kB$

where B (t) is the number of bacteria present at time t and k is the constant of proportionality.

In standard linear form the equation is $^{dB}/_{dt} - kB = 0$ and therefore P (t) = -k and f (t) = 0

The integrating factor is $I(t) = e^{\int -kdt} = e^{-kt}$

The differential equation becomes

$$\frac{d}{dt}(I(t)B) = I(t)\,f(t)$$

$$\text{i.e. } \frac{d}{dt}(e^{-kt}B) = 0$$

Integrating both sides gives

$e^{-kt}B = C$

Where C is a constant.

Rearranging to solve for B gives $B = Ce^{kt}$

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(You should recognise this formula from Further Integration, Unit 2 (7.4.3).)

Now we are given that there are initially 100 bacteria present in the culture. i.e.B = 100 when t = 0

Therefore $100 = Ce^0$ and so C = 100

Hence, the number of bacteria present at time t is given by

$B = 100e^{kt}$

2. A mixing problem

A tank contains 50 kg of salt dissolved in 200m^3 of water. Starting at time t = 0, water which contains 2 kg of salt per m^3, enters the tank at a rate of 4 m^3/minute and the well-stirred solution leaves the tank at the same rate. Calculate how much salt there is in the tank after 30 minutes.

Answer

Let M (t) represent the total mass of salt (in kg) in the tank at time t

Clearly the rate of change of mass of salt is equal to ${}^{dM}/_{dt}$

However, we can also calculate the rate of change of mass by calculating the rate at which the total amount of salt in the tank changes and so obtain a differential equation for M

The rate at which salt enters the tank is $2 \times 4 = 8$kg/min

The rate at which salt leaves the tank = mass of salt in the tank $\times$ proportion of solution which leaves the tank per minute

$= M \times {}^4/_{200} = 0.02$ per minute

Thus the rate of change of mass = 8 - 0.02 M kg/minute

Hence M (t) must satisfy the differential equation

${}^{dM}/_{dt} = 8 - 0.02M$

In standard linear form the equation is

${}^{dM}/_{dt} + 0.02M = 8$

and therefore we can identify P (t) = 0.02 and f (t) = 8

The integrating factor is $I(t) = e^{\int 0.02dt} = e^{0.02t}$

Therefore the differential equation becomes

$$\frac{d}{dt}[I(t)M] = I(t)f(t)$$

$$\frac{d}{dt}\left(e^{0.02t}M\right) = 8e^{0.02t}$$

Integrating both sides with respect to t gives

$$e^{0.02t}M = \frac{8}{0.02}e^{0.02t} + C$$

$$= 400e^{0.02t} + C$$

$$\Rightarrow M = 400 + Ce^{-0.02t}$$

Since there is initially 50 kg of salt in the tank, M (0) = 50 and so 50 = 400 + C i.e. C = -350

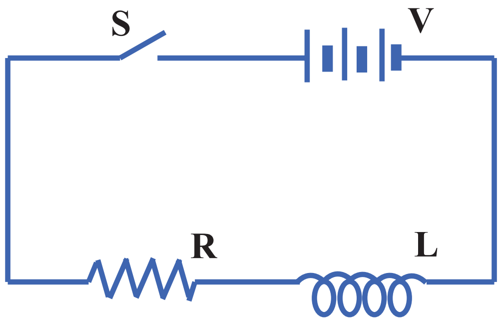
Hence our equation becomes $M = 400 - 350e^{-0.02t}$

When t = 30 then

$M = 400 - 350e^{-0.06} = 207.9$ kg

After 30 minutes there is 207.9 kg of salt in the tank.

3. An inductance-resistance circuit



This diagram represents an electrical circuit which contains a constant DC voltage source of 12 volts, a switch, a resistor of size 6 ohms, and an inductor of size L henrys. The switch is initially open but is closed at time t = 0 and the current begins to flow at that time. Let i (t) denote the current t seconds after the switch is closed. Then it is known that i (t) satisfies

$$\begin{cases} L\dfrac{di}{dt} + Ri = V \\ i\,(0) = 0 \end{cases}$$

Find a formula for i (t) in terms of L,R and V.

(Note that in this question L,R and V are constants.)

Answer

Rewrite the equation in standard linear form

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{V}{L}$$

and so we can easily see that $P\,(t) = {}^{R}/_{L}$ and $f\,(t) = {}^{V}/_{L}$

Calculate the integrating factor

$$I\,(t) = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

Therefore the differential equation becomes

$$\frac{d}{dt}\left(I\,(t)i\right) = I\,(t)\,f\,(t)$$

$$\frac{d}{dt}\left(e^{Rt/L}i\right) = \frac{V}{L}e^{Rt/L}$$

Integrating both sides we obtain

$$\int \frac{d}{dt}\left(e^{Rt/L}i\right)dt = \int \frac{V}{L}e^{Rt/L}dt$$

$$e^{Rt/L}i = \frac{V}{L}\frac{L}{R}e^{Rt/L} + C$$

$$= \frac{V}{R}e^{Rt/L} + C$$

We now substitute in the initial values and solve for y

When t = 0 then i = 0, and so we have

$$e^{o}.0 = \frac{V}{R}e^{0} + C$$

$$0 = \frac{V}{R} + C$$

$$\text{i.e.,} C = -\frac{V}{R}$$

Therefore

$$e^{Rt/L}i = \frac{V}{R}e^{Rt/L} - \frac{V}{R}$$

and so $i = \frac{V}{R} - \frac{V}{R}e^{-Rt/L}$

$$= \frac{V}{R}\left(1 - e^{-Rt/L}\right)$$

Now try the questions in Exercise 2.

Exercise 2

An on-line assessment is available at this point, which you might find helpful.

50 min

Q18: In the first few weeks after birth a baby gains weight at a rate proportional to its weight. A baby weighing 3.5 kg at birth weighs 4.2 kg after two weeks. How much did it weigh after 5 days (give your answer to two decimal places)?

Q19: Sugar dissolves in water at a rate proportional to the amount still undissolved. There was 80 kg of sugar initially and after 3 hours there is still 50 kg undissolved. How long will it take for half of the sugar to dissolve (give your answer to the nearest 5 minutes)?

Q20: A tank contains 1000 litres of salt water. 20 litres of fresh water flow into the tank and 20 litres of salt water flow out of the tank each minute.

Let M (t) represent the total amount of salt in the water at time t then the rate of change of salt in the tank can be described by the equation

$$\frac{dM}{dt} = -\frac{20}{1000}M$$

i.e. $\frac{dM}{dt} = -0.02M$

If the concentration of salt in the water was initially 30g/litre calculate the concentration of the salt in the water after 40 minutes. (Give your answer to 1 decimal place.)

Q21: A tank contains 100 kg of salt dissolved in 500m^3 of water. Starting at time t = 0, water which contains 3 kg of salt per m^3 enters the tank at a rate of 5 m^3/minute and the well-stirred solution leaves the tank at the same rate.

Let M (t) represent the total amount of salt in the water at time t then the rate of change of salt in the tank can be described by the equation

$$\frac{dM}{dt} = 15 - \frac{5}{500}M$$

i.e. $\frac{dM}{dt} = 15 - 0.01M$

a) Calculate how much salt there is in the tank after 1 hour.

b) How long does it take until there is 1000 kg of salt in the water?

c) What happens to the level of salt in the water as $t \Rightarrow \infty$?

Q22: An object of mass m is falling towards the earth. As it gets near the surface it is slowed down by air resistance proportional to its velocity so that, according to Newton's second law of motion

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$m^{dv}/_{dt} = mg - kv$

where v = v (t) is the velocity of the object at time t and g is the acceleration due to gravity near the surface of the earth. Assuming that the object falls from rest at time t = 0 i.e. v (0) = 0 then find an expression for v (t) for t $>$ 0

Q23: A patient is fed glucose intravenously at a constant rate. The change in the concentration of glucose in the blood with respect to time can be described by the differential equation

$\frac{dc}{dt} = \frac{G}{100V} - kc$

G, V and k are positive constants.

G is the rate at which glucose is administered, in milligrams per minute and V is the volume of blood in the body, in litres. The concentration c (t) is measured in milligrams per centilitre. The term -kx is included because it is assumed that the glucose is continually changing into other molecules at a rate propotional to its concentration.

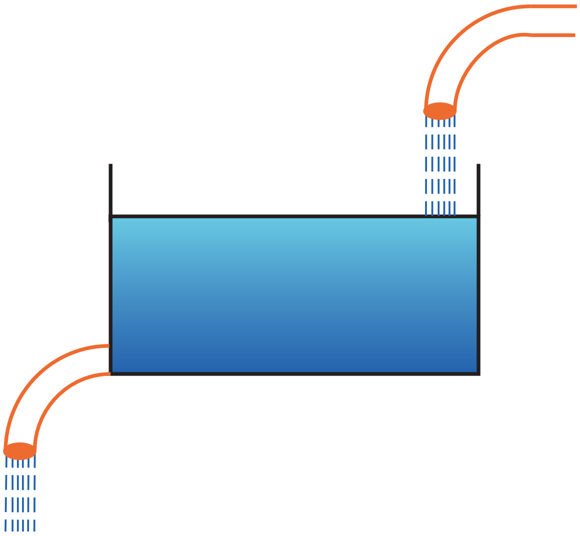
a) Solve the equation for c (t), given that c (0) = 100

b) Find the limit of the concentration of glucose in the blood as t $\Rightarrow \infty$

Save the fish

15 min

A rare species of fish is to be stored in a large tank containing 5000 litres of salt water. Initially the concentration of salt in the water is 20g/litre but the fish will survive best in a concentration of 12g/litre. Fresh water is poured into the tank and salt water is drained out at the same rate to reduce the concentration of salt in the water. What flow of water is needed for the concentration of salt in the water to reach the correct level after t minutes when the fish is first put in the water? Give your answer to the nearest whole number of litres.



There is a simulation on the course web site which allows you to check your answers for various values of t.

4.2 Second order, linear, differential equations with constant coefficients

4.2.1 Introduction

Learning Objective

Recognise second order, linear, differential equations

second order, linear, differential equation

A **second order, linear, differential equation** is an equation which can be expressed in the standard form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

where a, b and c are constants and $a \neq 0$.

This equation is second order because it contains a second derivative (but no derivatives of higher order).

non-homogeneous

A second order, linear, differential equation is **non-homogeneous** when it can be expressed in the standard form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

where a, b and c are constants and $a \neq 0$ and $f(x) \neq 0$

homogeneous

A second order, linear, differential equation is **homogeneous** when it can be expressed in the standard form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

where a, b and c are constants and $a \neq 0$

If an equation is second order we would expect it to be solved by carrying out two integrations and so we should also expect it to have a general solution containing two arbitrary constants.

In this topic we shall see how to find such solutions for second-order, linear, homogeneous and non-homogeneous differential equations with constant coefficients.

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4.3 Homogeneous, second order, linear, differential equations

Learning Objective

Solve homogeneous, second order, linear, differential equations with constant coefficients

We would like to find the general solution of

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

i.e. a solution containing two arbitrary constants.

Suppose that we could find two independent solutions y_1 and y_2 (i.e. y_1 is not a multiple of y_2)

Then it is easy to see that

1. $y_1 + y_2$ is also a solution
2. If A is any constant then Ay_1 is also a solution

It follows that if y_1 and y_2 are two independent solutions of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, then $Ay_1 + By_2$ is also a solution where A and B are arbitrary constants,

i.e. $Ay_1 + By_2$ is the **general solution** of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

Thus we can find the general solution provided we can find two independent solutions y_1 and y_2 for the equation.

We now see how to find these two distinct solutions.

The method used is to try and find appropriate values of m such that $y = e^{mx}$ is a solution.

If $y = e^{mx}$, then

$\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

Substituting the above into the second order linear differential equation we obtain

$$\begin{aligned} a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= 0 \\ \Leftrightarrow am^2e^{mx} + bme^{mx} + ce^{mx} &= 0 \\ \Leftrightarrow e^{mx}\left(am^2 + bm + c\right) &= 0 \end{aligned}$$

auxiliary equation

The second order, linear, homogeneous differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

has **auxiliary equation**

$am^2 + bm + c = 0$

Hence $y = e^{mx}$ is a solution of the differential equation if and only if m satisfies the quadratic equation

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$am^2 + bm + c = 0$

This quadratic equation, which is termed the auxiliary equation (or characteristic equation) of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, has roots given by

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

and these provide two independent solutions which enable us to find the general solution.

Three possible cases arise in solving the quadratic:

1. $b^2 - 4ac > 0$ when the quadratic has real distinct roots.
2. $b^2 - 4ac = 0$ when the quadratic has a repeated real root.
3. $b^2 - 4ac < 0$ when the quadratic has roots which are complex numbers.

We now see how the general solution can be found in each case.

4.3.1 Real and Distinct Roots

Learning Objective

Learn the form of the general solution when the roots of the auxiliary equation are real and distinct

If the roots are m_1 and m_2, then $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are independent solutions of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ and the general solution is

$y(x) = Ae^{m_1 x} + Be^{m_2 x}$

You can see how this works in the following example.

Example Find the general solution for the differential equation

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

Answer

The auxiliary equation for this equation is

$m^2 - 4m + 3 = 0$

We can factorise this to give

$(m - 1)(m - 3) = 0$

$\Rightarrow m = 1$ or $m = 3$

Therefore the roots of the auxiliary equation are $m_1 = 1$ and $m_2 = 3$

Hence the general solution of the differential equation is

$y = A\,e^{x} + B\,e^{3x}$

4.3.2 Equal Roots

Learning Objective

Learn the form of the general solution when the roots of the auxiliary equation are equal

If $b^2 - 4ac = 0$, the auxiliary equation has a single repeated root given by

$m = -{}^{b}/_{2a}$

Hence $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ has the solution $y_1 = e^{mx}$

A straightforward calculation shows that in this case $y_2 = xe^{mx}$ is also a solution.

(See Proof 1 for a complete explanation of this.)

Hence, in this situation, the equation has independent solutions e^{mx} and xe^{mx} and so has general solution

$y = A\,e^{mx} + Bx\,e^{mx}$

Example Find the general solution of the differential equation

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Answer

The auxiliary equation is

$m^2 - 4m + 4 = 0$

i.e.

$(m - 2)(m - 2) = 0$

$\Rightarrow m = 2$ (repeated root)

Hence the equation has independent solutions $y_1 = e^{2x}$ and $y_2 = xe^{2x}$

and therefore has general solution

$y = A\,e^{2x} + bx\,e^{2x}$

4.3.3 Complex Roots

Learning Objective

Learn the form of the general solution when the roots of the auxiliary equation are complex

If $b^2 - 4ac < 0$, the auxiliary equation has complex roots given by

$m_1 = \frac{-b-\sqrt{b^2 - 4ac}}{2a}$ and $m_2 = \frac{-b+\sqrt{b^2 - 4ac}}{2a}$

which can be written as the complex conjugate pair

$m_1 = p + iq$ amd $m_2 = p - iq$

The functions $e^{m_1 x}$ and $e^{m_2 x}$ now involve complex numbers and are not appropriate solutions of our differential equation. However, it is possible to deduce (see Proof 2) that in this case $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ has independent real solutions.

$y_1 = e^{px} \cos qx$ and $y_2 = e^{px} \sin qx$

Example Find the general solution for the differential equation

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

Answer

The auxiliary equation for the above equation is

$m^2 + 2m + 5 = 0$

This equation has roots given by the following

$$m = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Therefore p = -1 and q = 2 and the equation has independent solutions

$y_1 = e^{-x} \cos 2x$ and $y_2 = e^{-x} \sin 2x$

Hence the general solution is

$y = e^{-x} (A \cos 2x + B \sin 2x)$

We can summarise all of the above information as follows.

Strategy for solving

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

1. Write down the auxiliary equation

   $am^2 + bm + c = 0$

2. Solve the auxiliary equation by factorising or using the quadratic formula.

3. Write down the general solution, depending on the nature of the roots of the auxiliary equation, as follows:

   a) if roots m_1 and m_2 are real and distinct then the general solution is
   $y = Ae^{m_1x} + Be^{m_2x}$

   b) if there is a repeated root m then the general solution is
   $y = A\,e^{mx} + Bx\,e^{mx}$

   c) if there are complex roots $p \pm iq$ then the general solution is
   $y = e^{px}\,(A\cos qx + B\sin qx)$

You can now try the questions in Exercise 3.

50 min

Exercise 3

An on-line assessment is available at this point, which you might find helpful.

Find the general solutions of the following second order differential equations.

Q24: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

Q25: $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$

Q26: $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Q27: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Q28: $\frac{d^2y}{dx^2} = 10\frac{dy}{dx} - 25y$

Q29: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 0$

Q30: $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 3y$

Q31: $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

Q32: $9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = 0$

Q33: $2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

Q34: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = 0$

Q35:

In this question find the solutions for the auxiliary equation to two decimal places.

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$

4.3.4 Initial value problems

Learning Objective

Given initial conditions find the particular solution for a second order differential equation

If we are given a second order differential equation and initial conditions which must be satisfied by the solution we can find a **particular solution** of the equation satisfying the initial conditions.

As the general solution contains two arbitrary constants it is appropriate to specify two initial conditions.

Examples

1. Solve the initial value problem

$$\begin{cases} \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0 \\ y\ (0) = 2 \text{ and } y'\ (0) = 9 \end{cases}$$

Answer

The auxiliary equation for the differential equation is

$m^2 - 6m + 9 = 0$

$\Rightarrow (m - 3)^2 = 0$

$\Rightarrow m = 3$ (twice)

The roots of the above auxiliary equation are equal and so the general solution of the differential equation takes the form

$y\ (x) = (A + Bx)e^{mx} = (A + Bx)e^{3x}$

We can now find A and B to ensure that the initial conditions are satisfied.

Since y (0) = 2 we require

$(A + B \times 0)e^0 = 2$

$\Rightarrow A = 2$

Hence we must have $y\ (x) = (2 + Bx)e^{3x}$

We will now find B to ensure that $y'\ (0) = 9$. First we need to calculate $y'\ (x)$

$y\ (x) = (2 + Bx)e^{3x}$

$y'\ (x) = Be^{3x} + 3(2 + Bx)e^{3x}$ by the product rule

Thus we require

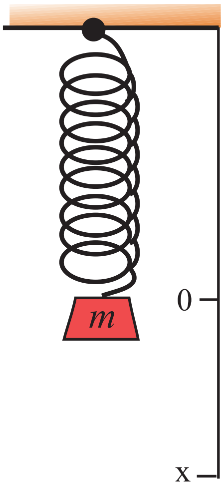
$9 = y'\ (0) = Be^0 + 3(2 + B \times 0)e^0 = B + 6$

$\Rightarrow B = 3$

Hence the solution for the initial value problem is

$y(x) = (2 + 3x)e^{3x}$

2. Simple harmonic motion



A mass is attached to a vertical spring and hangs in equilibrium. The mass is then pulled a distance of 10 cm down from its equilibrium position and released from rest so that it oscillates at the end of the spring.

If x (t) denotes the distance of the mass below its equilibrium position at time t, it can be shown that x (t) satisfies the differential equation

$\frac{d^2x}{dt^2} + 9x = 0$

(The equation $\frac{d^2x}{dt^2} + w^2x = 0$ is the equation of **simple harmonic motion**. There is further discussion of this equation in the Extended information chapter.)

Find x (t) and hence determine the position of the mass when t = 2

Answer

The differential equation

$\frac{d^2x}{dt^2} + 9x = 0$

has auxiliary equation $m^2 + 9 = 0$

which has solutions $m = \pm\sqrt{(-9)} = \pm 3i$

Obviously, we have complex roots of the form $p \pm iq$ where p = 0 and q = 3

Hence the equation has independent solutions $y_1 = e^{0t}\cos 3t = \cos 3t$ and $y_2 = e^{0t}\sin 3t = \sin 3t$ and so has general solution $x = A\cos 3t + B\sin 3t$

We are given that at t = 0

- the spring is 10cm from the equilibrium position i.e. x = 10 at t = 0
- the mass is at rest i.e. ${}^{dx}/_{dt} = 0$ at t = 0

We now choose A and B to obtain a particular solution which satisfies these initial conditions.

Since x = 10 when t = 0 we require

$10 = A\cos 0 + B\sin 0 \Rightarrow A = 10$

Hence $x(t) = 10\cos 3t + B\sin 3t$ and so $x'(t) = -30\sin 3t + 3B\cos 3t$

Since ${}^{dx}/_{dt} = 0$ when t = 0 we require

$0 = -30\sin 0 + 3B\cos 0 \Rightarrow 3B = 0 \Rightarrow B = 0$

Thus the position of the mass is given by

$x = 10\cos 3t$

Finally when t = 2 the position of the mass is $x = 10\cos 6 \approx 9.6$ cm below the equilibrium position.

Now try the questions in Exercise 4.

Exercise 4

60 min

An on-line assessment is available at this point, which you might find helpful.

Find the **particular solution** for each of the following differential equations with the given initial conditions.

Q36:

$$\begin{cases} \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \\ y\ (0) = 0 \text{ andy' } (0) = 10 \end{cases}$$

Q37:

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ with y = 0 and $^{dy}/_{dx} = 3$ when x = 0

Q38:

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ with y = 2 and $\frac{dy}{dx} = 3$ when x = 0

Q39:

$$\begin{cases} \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0 \\ y\ (1) = 7 \text{ and } y'\ (1) = 15 \end{cases}$$

Q40:

$$\begin{cases} \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0 \\ y(1) = 7 \text{ and } y'\ (1) = 30 \end{cases}$$

Q41:

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$ with y = 1 and $\frac{dy}{dx} = 11$ when $x = \pi$

Q42: A mass is attached to a vertical spring and hangs in equilibrium. At time t = 0 the mass is lowered 40 cm from its equilibrium position and given an initial velocity of 12 cm/second in the upward direction (i.e. $^{dx}/_{dt} = -12$).

If x (t) denotes the distance of the mass below its equilibrium position at time t, it can be shown that x (t) satisfies the differential equation

$\frac{d^2x}{dt^2} + 16x = 0$

Find x (t) and hence determine the position of the mass when t = 5

Q43: A mass is attached to a vertical spring which is immersed in a sticky fluid. At time t = 0 the mass is lowered 0.25 metres from its equilibrium position and given an initial velocity of 1m/second in the upward direction. (i.e. $^{dx}/_{dt} = -1$)

If x (t) denotes the distance of the mass below its equilibrium position at time t, it can be shown that x (t) satisfies the differential equation

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$

Find x (t)

Q44: The motion of a certain mass on the end of a spring is governed by the differential equation

$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + 0.125x = 0$

where x (t) denotes the distance in metres of the mass below its equilibrium position at time t. When t = 0 the mass is lowered 0.7 metres from its equilibrium position and is released from rest. Find x (t)

4.4 Non-homogeneous, second order, linear, differential equations

Learning Objective

Solve non-homogeneous, second order, linear, differential equations with constant coefficients

In this section we see how to find the general solution of the non-homogeneous (or inhomogeneous) equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

where f (x) is a simple function such as a polynomial, exponential or sine function.

As the equation is of second order the **general solution** should contain two arbitrary constants. We can find the general solution in the following way.

1. Find a general solution, y_c of the corresponding homogeneous equation

   $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

   complementary function

   The non-homogeneous equation

   $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

   has a corresponding homogeneous equation

   $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

   The homogenous equation has a general solution, y_c, which contains two constants.

   y_c is the **complementary function** for the non-homogeneous equation.

2. Find (by means of the method of undetermined coefficients, which you will learn about later) an 'obvious' special solution y_p of

   $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

   particular integral y_p is a **particular integral** of the differential equation

   $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

when y_p is a solution to the equation.

3. Write down the general solution of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ as

   $y = y_p + y_c$

   (This is because y contains two arbitrary constants and

   $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = a\frac{d^2y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P + a\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C$$
   $$= f(x) + 0 = f(x)$$

   so that y is a solution of the equation.)

Example a) Show that $y = e^{2x}$ is a particular integral of the differential equation

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4e^{2x}$

b) Hence find the general solution for this equation.

Answer

a) If $y = e^{2x}$ then

$\frac{dy}{dx} = 2e^{2x}$ and $\frac{d^2y}{dx^2} = 4e^{2x}$

Therefore

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4e^{2x} + 2e^{2x} - 2e^{2x} = 4e^{2x}$

and so $y_p = e^{2x}$ is a particular integral of the equation.

b) The corresponding homogeneous equation is

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

This has **auxiliary equation**

$m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2$ or $m = 1$

Thus the **complementary function** is

$y_c = A\,e^{-2x} + B\,e^x$

for A and B arbitrary constants.

Therefore the equation has general solution

$y = y_p + y_c$

$= e^{2x} + Ae^{-2x} + Be^x$

Thus we can use the following strategy to find the general solution of a non-homogeneous, second order, linear, differential equation.

Strategy for solving

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

1. Find the complementary function y_c
   i.e. find the general solution of the corresponding homogeneous equation
   $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$
2. Find a particular integral y_p
3. Write down the required general solution
   $y = y_p + y_c$

The difficult step in the method is finding the particular solution y_p. We shall see how this can be done in the next section but you may find it useful to first obtain some practice with the overall method in Exercise 5.

Exercise 5

15 min

An on-line assessment is available at this point, which you might find helpful.

Q45:

a) Show that $y = e^{4x}$ is a particular integral for the equation

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{4x}$

b) Hence find the general solution for this equation.

Q46:

a) Show that $y = e^{3x}$ is a particular integral for the equation

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 20e^{3x}$

b) Hence find the general solution for this equation.

4.4.1 Finding particular integrals

Learning Objective

Find particular integrals using the method of undetermined coefficients

We now see how to find particular integrals of the differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

by using the method of undetermined coefficients. This relies on the idea that for simple functions it is easy to guess the general form of a particular integral and then unknown coefficients in the general form can be found by substituting into the differential equation.

We will examine how this can be done in the cases where f (x) is a polynomial, where f (x) is an exponential and where f (x) is a sine or cosine function.

Polynomial functions

If f (x) is a polynomial of degree n, it is always possible to find a particular solution of the equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \quad (c \neq 0)$

of the form $y_P = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$

Example Find a particular integral for

$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 2x$

Answer

In this example the particular integral will have the form $y_p = ax + b$ where a and b are the undetermined coefficients.

We find values for a and b to ensure that y_p is a particular integral.

When $y_P = ax + b$ then $\frac{dy_P}{dx} = a$ and $\frac{d^2y_P}{dx^2} = 0$

Hence y_p is a solution, provided

$$\begin{aligned} \frac{d^2y_P}{dx^2} - \frac{dy_P}{dx} + 2y_P &= 2x \\ \text{i.e. } 0 - a + 2(ax + b) &= 2x \\ 2ax + (2b - a) &= 2x \end{aligned}$$

Equating coefficients we obtain

$2a = 2 \Rightarrow a = 1$

and

$2b - a = 0 \Rightarrow b = {}^a/_2 \Rightarrow b = {}^1/_2$

Thus we now have the particular integral

$y_p = x + {}^1/_2$

Note

- If f (x) is a quadratic e.g. $f(x) = 3x^2 - 4x + 1$ or $f(x) = x^2 - 5$ we would seek a particular solution of the form $y_p = ax^2 + bx + c$
- If f(x) is a constant e.g. f (x) = 3 we would seek a particular integral of the form f (x) = a

Exponential functions

It is usually possible to find a particular integral for the equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = e^{rx}$

of the form $y_p = ke^{rx}$ where k is the undetermined coefficient.

Example Find a particular integral for

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = e^{2x}$

Answer

We make a guess and try $y_p = k\,e^{2x}$ Then

$\frac{dy_P}{dx} = 2ke^{2x}$ and $\frac{d^2y_P}{dx^2} = 4ke^{2x}$

Hence y_p is a particular integral provided

$$\begin{aligned} 4ke^{2x} + 2\left(2ke^{2x}\right) - 3ke^{2x} &= e^{2x} \\ \text{i.e.}\,5ke^{2x} &= e^{2x} \\ 5k &= 1 \\ k &= \tfrac{1}{5} \end{aligned}$$

Thus the function $y_p = {}^1/_5 e^{2x}$ is a particular solution for the differential equation.

Note

The above method for finding a particular solution of

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = e^{rx}$

fails when r coincides with one of the roots of the auxiliary equation $am^2 + bm + c = 0$

In this case we can still find a particular integral of the form:

- $y_p = kx\,e^{rx}$ if r is a single root of the auxiliary equation;
- $y_p = kx^2\,e^{rx}$ if r is a repeated root of the auxiliary equation.

Sine and Cosine functions

It is always possible to find a particular integral of the equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \sin nx, (b \neq 0)$

or of

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \cos nx, (b \neq 0)$

of the form

$y_p = p \sin nx + q \cos nx$

where p and q are the undetermined coefficients.

It would have been much more convenient if the particular integrals could have been respectively $y_p = p \sin nx$ and $y_p = q \cos nx$. Unfortunately, this simpler and more obvious approach does not work and the general form $y_p = p \sin nx + q \cos nx$ should always be used.

Example Find a particular solution for the differential equation

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5 \sin x$

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Answer

We try a particular solution of the form $y_p = p \sin x + q \cos x$

Hence

$\frac{dy_P}{dx} = p \cos x - q \sin x$ and $\frac{d^2y_P}{dx^2} = - p \sin x - q \cos x$

Hence y_p is a particular solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5 \sin x$ provided

$$
\begin{aligned}
- p \sin x - q \cos x + 2(p \cos x - q \sin x) + 2(p \sin x + q \cos x) &= 5 \sin x \\
\text{i.e. } (p - 2q)\sin x + (2p + q)\cos x &= 5 \sin x \\
\text{i.e. } p - 2q = 5 \text{ and } 2p + q &= 0 \\
\text{i.e. } p = 1 \text{ and } q &= - 2
\end{aligned}
$$

The particular solution is therefore

$y_p = \sin x - 2 \cos x$

Note

For the differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

if f (x) is a sum of terms i.e. f (x) = p (x) + q (x) and

y = u (x) is a particular integral of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = p(x)$ and

y = v (x) is a particular integral of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = q(x)$

then it can be shown that y = u (x) + v (x) is a particular integral of

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = p(x) + q(x)$

Now try the questions in Exercise 6.

Exercise 6

50 min

An on-line assessment is available at this point, which you might find helpful.

In the following questions find

a) the **complementary function**, y_c

b) a **particular solution**, y_p

c) the **general solution**

of the non-homogeneous equation.

Q47: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{4x}$

Q48: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x^2 + 3$

Q49: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 25 \sin x$

Q50: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 8 - 10x$

Q51: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 18$

Q52: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 10e^x$

Q53: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 21e^{2x}$

Q54: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y + 50 \cos 3x = 0$

Q55: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3 \sin x + 19 \cos x$

Q56: Find the general solution of the differential equation

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = f(x)$
in each of the cases

a) $f(x) = 12x$
b) $f(x) = 8e^{3x}$
c) Hence find the general solution of
   $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12x + 8e^{3x}$

4.5 Summary

Learning Objective

Recall the main learning points from this topic

A. First order linear differential equations can be solved in the following way

1. Write the equation in standard linear form
   $\frac{dy}{dx} + P(x)\, y = f(x)$
2. Calculate the integrating factor
   $I(x) = e^{\int p(x)\, dx}$
3. Multiply the equation in standard linear form by I (x) and obtain the equation
   $\frac{d}{dx}[I(x)\, y] = I(x)\, f(x)$
4. Integrate both sides of this last equation to give
   $I(x)\, y = \int I(x)\, f(x)\, dx$
5. Rearranged to solve for y

B. When an **initial condition** is given for a first order, linear, differential equation we should then be able to find a **particular solution**.

C. The homogeneous, second order, linear differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

can be solved in the following way

1. Write down the auxiliary equation

   $am^2 + bm + c = 0$

2. Solve the auxiliary equation by factorising or using the quadratic formula.

3. Write down the general solution depending on the nature of the roots of the auxiliary equation as follows

   a) if roots m_1 and m_2 are real and distinct then the general solution is
      $y = Ae^{m_1 x} + Be^{m_2 x}$

   b) if there is a repeated root m then the general solution is
      $y = A e^{mx} + Bx e^{mx}$

   c) if there are complex roots $p \pm iq$ then the general solution is
      $y = e^{px} (A \cos qx + B \sin qx)$

D. When we are given two initial conditions we can find a **particular solution** of a second order linear differential equation satisfying these initial conditions.

E. The non-homogeneous, second order, linear differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

can be solved in the following way

a) Find the complementary function y_c

   i.e. find the general solution of the corresponding homogeneous equation

   $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

b) Find a particular integral y_p

c) Write down the required general solution

   $y = y_p + y_c$

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F. In certain cases it is possible to find a **particular integral** for the equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \quad (c \neq 0)$

1. If f (x) is a polynomial of degree n then the particular integral is of the form $y_P = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$
2. If $f(x) = e^{rx}$ then the particular integral is usually of the form $y_p = k\,e^{rx}$
3. If f (x) = sin nx or f (x) = cos nx then the particular integral is usually of the form $y_p = p \sin nx + q \cos nx$

4.6 Proofs

Proof 1

For a homogeneous, second order, linear, differential equation with equal roots show that when $y_1 = e^{mx}$ is a solution then $y_2 = xe^{mx}$ is also a solution.

Proof

The differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

has auxiliary equation

$am^2 + bm + c = 0$

The auxiliary equation has equal roots when $b^2 - 4ac = 0$ and from the quadratic formula

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

the repeated root is given by $m = -{}^{b}/_{2a}$

We saw in section 14.3 that $y_1 = e^{mx}$ is one solution of the homogeneous equation. In the case where m is a repeated root we try $y_2 = xe^{mx}$ as another solution.

Then $\frac{dy_2}{dx} = mx\,e^{mx} + e^{mx}$ and $\frac{d^2y_2}{dx^2} = m^2x\,e^{mx} + 2m\,e^{mx}$

Hence

$$\begin{aligned} a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2 &= am^2xe^{mx} + 2ame^{mx} + bmxe^{mx} + be^{mx} + cxe^{mx} \\ &= \left(am^2 + bm + c\right)xe^{mx} + (2am + b)e^{mx} \\ &= 0 \quad \left(\text{since } am^2 + bm + c = 0 \text{ and } m = -\frac{b}{2a}\right) \end{aligned}$$

Therefore, if y is a repeated root of the auxiliary equation, the differential equation has solutions $y_1 = e^{mx}$ and $y_2 = xe^{mx}$ and we can write the general solution as

$y = A\ e^{mx} + Bx\ e^{mx}$

Proof 2

Show that the homogeneous, second order, linear, differential equation with complex roots has independent real solutions, $y_1 = e^{px}\cos qx$ and $y_2 = e^{px}\sin qx$.

Proof

The differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

has auxiliary equation

$am^2 + bm + c = 0$

If $b^2 - 4ac < 0$ then the auxiliary equation has complex roots which can be written as $m = p \pm iq$

Now if $m = p \pm iq$ is a solution to the auxiliary equation then $y = e^{mx} = e^{(p + iq)x}$ is a solution of the differential equation. Unfortunately y involves complex numbers and we would like to find solutions of the equation invoving only real numbers.

But

$$\begin{aligned} y = e^{(p+iq)x} &= e^{px}e^{iqx} \\ &= e^{px}\left(e^{ix}\right)^q \\ &= e^{px}(\cos x + i\sin x)^q \quad \text{(see Further sequences and series)} \\ &= e^{px}(\cos qx + i\sin qx) \quad \text{(by De Moivre's theorem)} \end{aligned}$$

It can be checked that both the real and imaginary parts of y give rise to solutions of the differential equation and hence we obtain the independent solutions $y_1 = e^{px}\cos qx$ and $y_2 = e^{px}\sin qx$ and we can write the general solution as

$y = e^{px}(A\cos qx + B\sin qx)$

4.7 Extended information

There are links on the course web site to a variety of web sites related to this topic. These include references to The Tacoma Bridge Disaster of 1940, an example of where harmonic motion went wrong. The Millenium Bridge in London has also had oscillation problems and we include a site giving interesting background information. There is also a site which can take you on to more in-depth study of differential equations, if you wish, and there are two sites which demonstrate harmonic motion.

Leonhard Euler (1707 - 1783)

Leonhard Euler was born on the 15th April 1707 in Basel, Switzerland.

While living with his grandmother, he went to school in Basel but learned no mathematics at all there. His interest in mathematics was first sparked by his father, Paul, and indeed Euler read mathematics texts on his own and even took some private lessons.

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He entered the University of Basel in 1720 at the age of 14 and in 1723 he completed his Master's degree having compared and contrasted the philosophical ideas of Descartes and Newton. He then went on to study mathematics and completed his studies in 1726. In 1727 he submitted an entry for the Grand Prize of the Paris Academy on the best arrangement of masts on a ship. His essay won him second place which was an excellent achievement for a young graduate. In later life he did go on to win this prize on two occasions, in 1738 and 1740.

He served as a medical lieutenant in the Russian navy from 1727 to 1730. However, when he became professor of physics at the St Petersburg Academy he was able to give up his Russian navy post.

On 7th January 1734 he married Katharina Gsell who was from a Swiss family and the daughter of a painter. They had 13 children altogether, although only five survived their infancy. It is said that Euler claimed to have made some of his greatest mathematical discoveries while holding a baby in his arms while others played around his feet.

Euler is best known for his analytical treatment of mathematics and his discussion of calculus concepts, but he is also credited for work in acoustics, mechanics, astronomy, and optics. Indeed his work in mathematics is so vast that he is considered the most prolific writer of mathematics of all time.

He discovered the procedure for solving linear differential equations with constant coefficiants in1739. This arose out of his work on various problems in dynamics that led to differential equations of this type.

We owe to Euler the notation f (x) for a function, i for $\sqrt{}$(-1), π for pi, σ for summation and many others.

In 1735 Euler had a severe fever and almost lost his life. In1738 he started to have problems with his eyesight and by 1740 he had lost the use of an eye. By 1766 he became almost entirely blind after an illness. Incredibly after the age of 59, when he was now totally blind, he produced almost half of his total works. Upon losing the sight in his right eye he is quoted as saying '*Now I will have less distraction.*'

Euler died in 1783. His last day has been described as follows.

On 18 September 1783 Euler spent the first half of the day as usual. He gave a mathematics lesson to one of his grandchildren, did some calculations with chalk on two boards on the motion of balloons; then discussed with Lexell and Fuss the recently discovered planet Uranus. About five o'clock in the afternoon he suffered a brain haemorrhage and uttered only '*I am dying*' before he lost consciousness. He died at about eleven o'clock in the evening.

After his death the St Petersburg Academy continued to publish Euler's unpublished work for nearly 50 more years.

Robert Hooke

Robert Hooke was born on the 18th July 1707 in Freshwater, Isle of Wight, and died on 3rd March 1703 in London, England.

He went to school in Westminster where he learned Latin and Greek and in 1653 he went to Christ College, Oxford.

In 1660 he discovered the law of elasticity, known as Hooke's law. He worked on optics, simple harmonic motion and stress in stretched springs. He applied these studies in his designs for the balance of springs in watches.

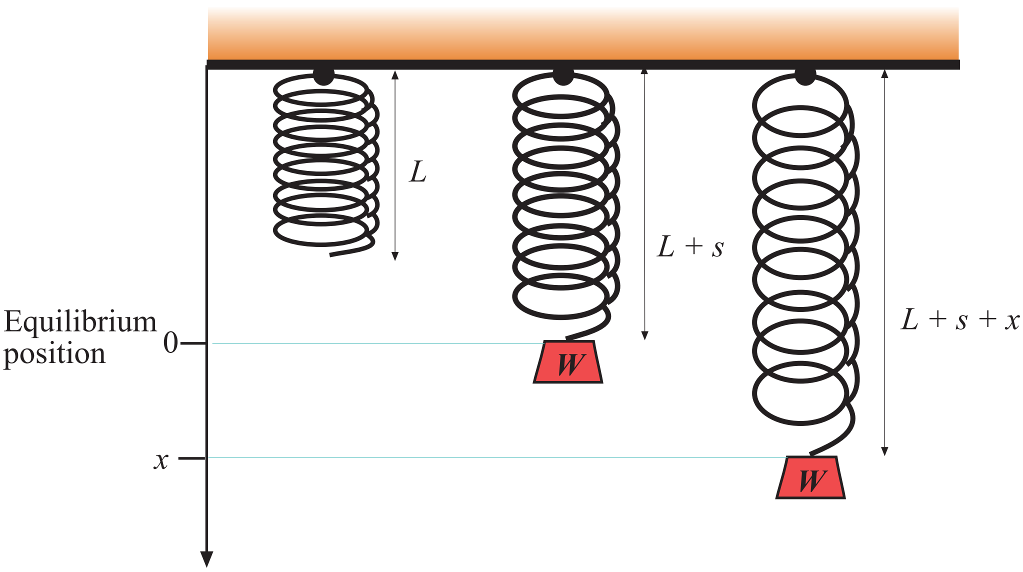
In 1662 he was appointed curator of experiments to the Royal Society of London and was elected a fellow the following year. In 1665 he became professor of geometry at Gresham College, London where he remained for 30 years. In addition to this he also held the post of City Surveyor and was a very competent architect. He was chief assistant to Sir Christopher Wren in his project to rebuild London after the great fire of 1666.

In 1665 he first achieved world wide fame after the publication of his book *Micrographia* ("Small Drawings"). This book contained beautiful pictures of objects Hooke had studied through a microscope that he made himself, including the crystal structure of snowflakes. His studies of microscopic fossils led him to become one of the first proposers of a theory of evolution.

There is no portrait of Hooke that is known to exist. Perhaps because he has been described as a *'lean, bent and ugly man'* and so was maybe reluctant to sit and have his portrait painted.

Simple harmonic motion

One of the most useful applications of second order differential equations is in the study of vibrations or oscillations. This type of motion occurs in many situations but the classic example is the motion of a spring.



Consider a weight of mass m, suspended from a spring of natural length L. The weight will stretch the spring to a new length, L + s

Hooke's Law tells us that the tension in the spring acts back up towards the equilibrium position and is proportional to the amount the spring has stretched. Thus the upward force exerted by the spring is ks and since the mass is in equilibrium this is balanced by the force of gravity mg acting downwards on the mass. Thus we have

ks = mg

Suppose that the spring is pulled downwards by an additional amount x (positive

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direction downwards) from this equilibrium position. Then the spring has been stretched by a total of s + x and so the spring now exerts a force of k(s + x) upwards. Thus the net downward force acting on the spring is

mg - k(s + x) = mg - ks - kx = -kx

But by Newton's Second Law

mass $\times$ acceleration = force

and so we obtain the differential equation

$m\frac{d^2x}{dt^2} = -kx$

Dividing this equation by m and rearranging, we obtain the equation

$\frac{d^2x}{dt^2} + w^2x = 0$, where $w^2 = \frac{k}{m}$

equation of simple harmonic motion The equation

$\frac{d^2x}{dt^2} + w^2x = 0$

is the **equation of simple harmonic motion**.

It has general solution

x = A cos wt + B sin wt

This second order differential equation has auxiliary equation $m^2 + w^2 = 0$ which has solutions

$m = \pm\sqrt{(-w^2)} = \pm wi$

Obviously, this has complex roots of the form $p \pm qi$ where p = 0 and q = w. Hence the general solution is

$$\begin{aligned} x &= e^{0t}(A \cos wt + B \sin wt) \\ &= A \cos wt + B \sin wt \end{aligned}$$

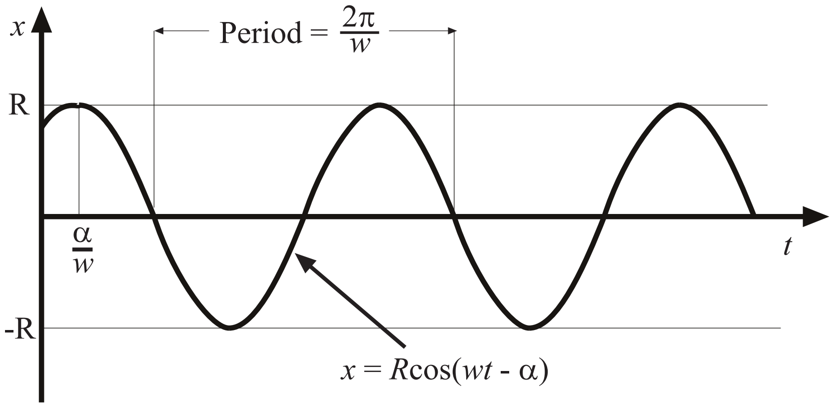
This can be rewritten in the alternative form

$y = R\cos(wt - \alpha)$

with $R = \sqrt{A^2 + B^2}$ and $\alpha = \tan^{-1}\left(\frac{B}{A}\right)$

This equation represents simple harmonic motion of amplitude R and period $2\pi/w$ The angle α is the phase angle.

The graph for this solution is shown here



This graph represents the graph of simple harmonic motion of a frictionless spring. It is referred to as undamped vibration.

Damped harmonic motion

If the motion of the spring is retarded by a friction force, which is proportional to the velocity, then the equation of motion becomes

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

$c^{dx}/_{dt}$ is the friction force with c the constant of proportionality.

This equation can then be rewritten as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + w^2x = 0, \text{where } 2b = \frac{c}{m} \text{ and } w^2 = \frac{k}{m}$$

The auxiliary equation is $m^2 + 2bm + w^2 = 0$ which has roots

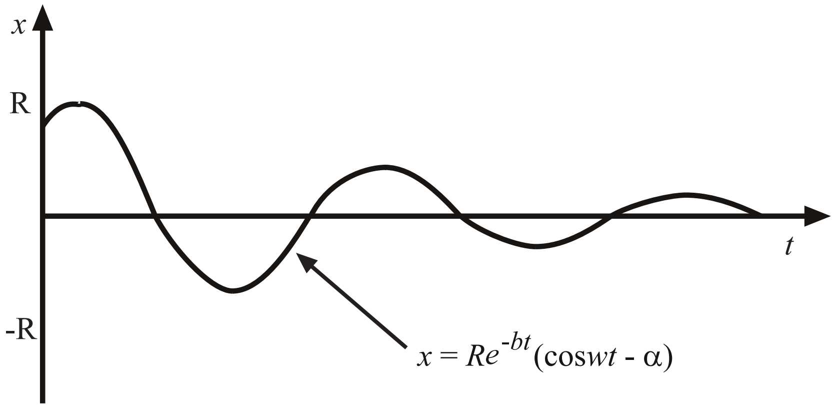
$$m = -b \pm \sqrt{(b^2 - w^2)}$$

If $b < w$ then the differential equation has complex roots $-b \pm qi$, $\left(q = \sqrt{w^2 - b^2}\right)$, and the general solution takes the form

$$\begin{aligned} x &= e^{-bt}(A \cos qt + B \sin qt) \\ &= e^{-bt}R(\cos qt - \alpha) \end{aligned}$$

As you can see, this is similar to the equation for undamped harmonic motion. However, due to the factor e^{-bt} the amplitude diminishes exponentially as $t \Rightarrow \infty$ because $e^{-bt} \Rightarrow 0$ as $t \Rightarrow \infty$

The graph for $x = e^{-bt} R (\cos wt - \alpha)$ follows. This solution represents damped vibratory motion as the result of friction. Notice that the vibrations tend to die out as time progresses.



4.8 Review exercise

Review exercise in further ordinary differential equations

15 min

An on-line assessment is available at this point, which you might find useful.

Q57: Obtain the general solution of the first-order linear differential equation

${}^{dy}/_{dx} = 5y = e^{-x}$

Q58: Obtain the general solution of the first-order linear differential equation

$\frac{dy}{dx} + 2xy = e^{-x^2} \cos x$

4.9 Advanced review exercise

Advanced review exercise in further ordinary differential equations

20 min

An on-line assessment is available at this point, which you might find helpful.

Q59: Find the particular solution of the initial value problem

$x\frac{dy}{dx} + 2y = 6x^4$, x = 1 when y = 5

Q60:

a) Show that the differential equation
$\frac{dy}{dx} + (\tan x)\, y = \cos^2 x$
has integrating factor $I(x) = \frac{1}{\cos x}$

b) Hence find a solution of the differential equation for which $y = {}^3/_2$ when $x = \pi/4$

Q61: Find the general solution of the differential equation

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$

Q62: Find the particular solution of the differential equation with the given initial conditions

$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$, $y(0) = 3$, $y'(0) = 20$

Q63: Find the solution of the differential equation

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = f(x)$

in each of the cases

a) $f(x) = 40 \cos x$
b) $f(x) = 40 \sin x$
c) $f(x) = 40 \cos x + 40 \sin x$

4.10 Set review exercise

Set review exercise in further ordinary differential equations

15 min

An on-line assessment is available at this point, which you will need to attempt to have these answers marked. These questions are not randomised on the web site. The questions on the web may be posed in a different manner but you should have the required answers in your working.

Q64: Obtain the general solution of the first-order linear differential equation

$^{dy}/_{dx} = {}^{y}/_{x} = 4x^2$

Q65: Obtain the general solution of the first-order linear differential equation

$\frac{dy}{dx} + 3x^2y = (2x + 1)e^{-x^3}$

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Topic 5

Further number theory and further methods of proof

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Learning Objectives

- *Use further number theory and direct methods of proof.*

Minimum Performance Criteria:

- *Use proof by mathematical induction.*
- *Use Euclid's algorithm to find the greatest common divisor of two positive integers.*

Prerequisites

A sound knowledge of the following subjects is required for this unit:

- Algebraic manipulation.
- Basic number theory concepts.
- Standard results involving summations.

5.1 Revision

If any question in this revision exercise causes difficulty then it may be necessary to revise the techniques before starting the topic. Ask a tutor for advice.

Q1: Explain the relationship between the terms 'theorem', 'conjecture' and 'proof'.

Q2: Find S_4 for $\sum_{r=1}^{\infty}\left(r + \frac{3}{r}\right)$

Q3: If $(x^2 + 3y)\left(4x + \frac{5}{y}\right) = 5x^4\left(\frac{4}{5x} + \frac{1}{x^2y}\right)$, show that $xy = -\frac{5}{4}$

Q4: State the meaning of the symbols:

1. $\Rightarrow$
2. $\Leftarrow$
3. $\Leftrightarrow$

5.2 Implications

Learning Objective

Recognise and use implications on propositions.

Mathematical propositions by their nature are either **true** or **false**. They must have one truth value (either 1 for true or 0 for false) and cannot have both.

Example Consider the proposition 2 + 3 = 5; this is a true proposition.

On the other hand 2 + 3 = 3 is a false proposition.

In number theory, the truth of a proposition has a bearing on the symbolism used.

Many of the symbols and terms that are important in mathematics come from the fascinating topic called mathematical logic, which is not part of this course.

Necessary and sufficient conditions

In topic 10 the following definitions of 'implies' and 'implied by' were given.

'**Implies** means that the first statement can be used to logically deduce the next statement. It is denoted by the symbol $\Rightarrow$'.

'**Implied by** means that the first statement is a logical consequence of the second statement. It is denoted by the symbol $\Leftarrow$'.

The expression P $\Rightarrow$ Q reads 'P implies Q'.

P is called the **antecedent**, the **premise** or the **hypothesis**.

Q is the **conclusion** or **consequent**.

There are however many other precise ways of saying exactly the same thing. The following statements are all ways of saying that P $\Rightarrow$ Q:

- P implies Q
- if P then Q
- Q follows from P
- Q if P
- Q is necessary for P
- P is sufficient for Q

Also P $\Leftarrow$ Q, meaning P is implied by Q, has a similar list:

- Q implies P
- if Q then P
- P follows from Q
- P if Q
- P is necessary for Q
- Q is sufficient for P

These two lists demonstrate clearly that the symbol $\Leftarrow$ gives the reverse implication of the symbol $\Rightarrow$

The remainder of this section will concentrate on the $\Rightarrow$ sign but clearly the same explanations can be found for the implication $\Leftarrow$ (which is much less commonly used).

The exercise which follows is similar, but not identical, to that given in topic 10. It is still worthwhile taking a few extra minutes to complete it at this stage.

5 min

Symbol exercise

Learning Objective

Use the correct symbol

There is an alternative version of this exercise on the web for you to try if you wish.

Q5: Insert the correct symbol between proposition A and proposition B in the following:

1. A: The liquid is colourless and clear.
   B: The liquid is drinking water.
2. A: $x^2 = -2$
   B: x is a complex number.
3. A: R is a square matrix.
   B: The inverse matrix R^{-1}exists.
4. A: n is an odd prime.
   B: n is an odd number.

Q6: Reword the following implications using either P$\Rightarrow$ Q or P $\Leftarrow$ Q

1. If x + 8 = 4 then x is a negative number.
2. $x \geq 0$ is sufficient for x + 1 to be positive.
3. sin $\theta = 0$ is necessary for $\theta = \pi$

The two phrases 'Q is necessary for P' and 'P is sufficient for Q', both of which mean P $\Rightarrow$ Q, give a different emphasis to the propositions.

Look at a truth table for P $\Rightarrow$ Q. The table is constructed by labelling the columns with the propositions P and Q and the implication P $\Rightarrow$ Q. The rows are then completed by considering the combinations of true (T = 1) and false (F = 0) which can occur. (T = 1 and F = 0 are known as Boolean constants.)

P	**Q**	**P $\Rightarrow$ Q**
1	1	1
1	0	0
0	0	1
0	1	1

The last two lines may seem puzzling, but can be explained by considering the meaning of P $\Rightarrow$ Q as 'if P then Q'. Hence if P is true, Q follows by some logical deduction: but if P is **not** true, Q might still be true for some other reason.

Examples

1. If P is the statement 'it is Saturday' and Q is the statement 'it is the weekend' then P $\Rightarrow$ Q. But statement Q is true on Sundays too, when P is false.

2. If P is the statement '$x > 3$' and Q is the statement '$x^2 > 9$' then P $\Rightarrow$ Q. But if, for example, x = -4, then P is false, but Q is true.

So when P $\Rightarrow$ Q either P is true - and then Q must be true - or P is false and no conclusion can be made about Q.

Hence P $\Rightarrow$ Q is logically equivalent to the statement 'either P is false, or Q is true, or both' and when P is false, the implication P $\Rightarrow$ Q is true.

This may seem at odds with the usual sense of the word 'implication' but it is correct within the framework of mathematical reasoning.

3. If P is the statement 'x is an odd number and $1 < x < 9$' and Q is the statement 'x is a prime number' then P $\Rightarrow$ Q.

If P is false, then Q can be either true - for example when x = 13 - or false as when x = 4 or x = 14.

However as a single mathematical proposition P $\Rightarrow$ Q is true.

To say 'P is sufficient for Q' indicates that if P is false the implication is true regardless of P but that if P is true then Q has to be true for the implication to hold.

P is termed the **sufficient condition**.

Sufficient condition

If P $\Rightarrow$ Q then P is a sufficient condition for Q.

The phrase 'Q is necessary for P' however, indicates that Q has to be true if P is true for the implication to hold but Q can be true independently of P and give a true implication.

Q is termed the **necessary condition**.

Necessary condition

If P $\Rightarrow$ Q then Q is a necessary condition for P.

The same concepts apply to the lists of related phrases for P $\Leftarrow$ Q which were shown earlier.

Bi-implication or equivalence The implication P $\Leftrightarrow$ Q can be read as:

- P if and only if Q
- P is equivalent to Q
- P implies Q and Q implies P

The definition as given in topic 10 follows.

Equivalent to

Is equivalent to means that the first statement implies and is implied by the second statement. (The first statement is true **if and only if** the second statement is true.)

The truth table for $\Leftrightarrow$ is

P	**Q**	**P $\Leftrightarrow$ Q**
1	1	1
1	0	0
0	0	1
0	1	0

Example Let P be the statement:

A triangle has side lengths a, b and c (with c the largest) and contains a right angle.

Let Q be the statement:

A triangle has side lengths a, b, c (with c the largest) and $a^2 + b^2 = c^2$

Then P $\Leftrightarrow$ Q

5 min

Equivalence exercise

There is a short exercise similar to this on the web for you to try if you prefer it.

Q7: State, with explanation, which of the following statements is correct with the equivalence symbol of $\Leftrightarrow$ replacing the question mark.

1. In the Smith family, Joe is Amy's father ? Amy is Joe's daughter.
2. x is divisible by 6 ? x is an even number.
3. I study Maths ? I am good at Maths.
4. x is an odd number ? x is a prime > 2

Negation

The symbol $\neg$ is used to denote negation in this topic. There are alternative symbols such as - and ~ . The definition of negation is as follows.

Negation

- If the proposition P is true then its negation $\neg$P is false.
- If the proposition P is false then its negation $\neg$P is true.

Both propositions P and $\neg$P cannot be true at the same time.

Q8: Complete the truth table for negation.

P	**$\neg$P**
1	
0	

Example State the negation of the proposition P : $x > 2$

Answer:

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The proposition is P and

the negation is ¬P : $x \leq 2$

This can be checked as follows:

Let x = 4 then P is true and ¬P is false.

Let x = 1 then P is false and ¬P is true.

Negation exercise

There is another small exercise on the web for you to try if you wish.

5 min

Q9: For each of the propositions P, state its negation ¬P.

1. P: $\sqrt{x}$ is rational.
2. P: I like Maths.
3. P: x is a negative real.
4. P: $x > 0$, $x \in \mathbb{Z}$.

Inverse and converse

Returning to the original implication $P \Rightarrow Q$, the inverse implication applies to the negation of both propositions P and Q.

Inverse

The inverse of the statement $P \Rightarrow Q$ is $\neg P \Rightarrow \neg Q$

The converse is also worth mentioning at this point.

Converse

The converse of the statement $P \Rightarrow Q$ is $Q \Rightarrow P$

Look at the truth tables of both.

Inverse

P	**¬P**	Q	**¬Q**	**¬P ⇒ ¬Q**
1	**0**	1	**0**	**1**
1	**0**	0	**1**	**1**
0	**1**	0	**1**	**1**
0	**1**	1	**0**	**0**

Converse

Q	P	Q ⇒ P
1	1	1
0	1	1
0	0	1
1	0	0

Notice that the results of the two truth tables are the same.

That is, $\neg P \Rightarrow \neg Q$ is logically the same as $Q \Rightarrow P$:

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the inverse is the logical equivalent of the converse.

Example State the converse of the implication $P \Rightarrow Q$ where P is the proposition $^1/_x \in \mathbb{R}$ and

Q is the proposition $x \neq 0$

Answer:

The implication as it stands reads 'if $^1/_x \in \mathbb{R}$ then $x \neq 0$'.

The converse is $Q \Rightarrow P$ and reads ' If $x \neq 0$ then $^1/_x \in \mathbb{R}$'.

Note that the converse implication does not need to hold.

5 min

Converse exercise

There is a similar web exercise for you to try if you wish.

Q10: Consider the implication S: 'if it is Thursday then I will be at home' and state the converse.

Q11: State the converse of the following implications:

1. I am over fifty $\Rightarrow$ I am over forty.
2. If it rains then it is cloudy.
3. I work only if I am at Heriot Watt University.
4. Joyce cycles to school if it is dry.

Contrapositive

The final implication to consider is the contrapositive.

It is the inverse of the converse.

Since the converse and the inverse are logically equivalent, from this alone it can be deduced that the contrapositive and the original implication are logically equivalent. (It is similar to taking the inverse of an inverse in matrix algebra or reflection twice in an axis in functions).

Contrapositive

The contrapositive of the implication $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$

Activity

Complete, compare and confirm that the truth tables for the implications $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are the same.

'When a philosopher says something which is true then it is trivial. When he says something that is not trivial then it is false.'

Carl Gauss 1777 - 1855

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5.3 Further proof by contradiction

Learning Objective

Apply the method of proof by contradiction

The following definition may be familiar.

Proof by contradiction

Proof by contradiction assumes the conjecture to be false and shows that this leads to the deduction of a false conclusion.

Apart from assuming that the original conjecture is false, the strategy in proof by contradiction depends, as do other types of proof, on exactly what the conjecture states.

Many proofs by contradiction can use the odd and even integer forms.

For example any even integer m takes the form m = 2s for some other integer s (since all even integers are divisible by 2).

An odd integer k takes the form k = 2t + 1 for some other integer t

Further details on this and on other integer forms will be given in the section on the division algorithm and more of these integer forms are shown in a separate section called Divisibility rules and integer forms.

Here is an example of a proof by contradiction using the odd and even forms of integers.

Example If m^2 is even then m is even. Prove this conjecture using the proof by contradiction method.

Answer:

Assume that the conjecture is false. So assume that m^2 is even but that m is odd.

Let m = 2t + 1 where t is an integer.

Then $m^2 = (2t + 1)^2 = 4t^2 + 4t + 1 = 2\ (2t^2 + 2t) + 1 = 2s + 1$ where $s = 2t^2 + 2t$

This is in the form of an odd integer and contradicts the assumption that m^2 is even.

Thus the conjecture is true.

Sometimes straightforward algebraic manipulation is needed, as in the next example.

Example Prove, by contradiction, that $\sqrt{3} + \sqrt{5} < \sqrt{17}$

Answer:

Assume that the conjecture is false

Make a new assumption that $\sqrt{3} + \sqrt{5} \geq \sqrt{17}$

Square both sides to give

$(\sqrt{3} + \sqrt{5})^2 = 3 + 2\sqrt{15} + 5 = 8 + 2\sqrt{15}$

$(\sqrt{17})^2 = 17$

$8 + 2\sqrt{15} \geq 17$

$\Rightarrow 2\sqrt{15} \geq 9$

Square both sides again to give

$60 \geq 81$ which is a contradiction.

Thus the assumption that $\sqrt{3} + \sqrt{5} \geq 17$ is untrue and the original conjecture is true.

That is, $\sqrt{3} + \sqrt{5} < \sqrt{17}$

In an example such as the last one it is important to realise that 'prove' means provide a reasoned logical argument. Calculation to show that the result holds is not a proof (though the proof *is* a calculation).

Divisibility rules are also useful. For example, an integer which is not a multiple of three has a remainder of 1 or 2 if divided by 3. This is the basis of the next example.

Example Prove, by contradiction, that for any integer n, if n^2 is a multiple of 3 then n itself is a multiple of 3

Answer:

Assume that the conjecture is untrue and make a new assumption that n^2 is a multiple of 3 but n is not a multiple of 3. There are therefore two possibilities to consider

1. n is of the form 3k + 1 for some integer k
2. n is of the form 3k + 2 for some integer k

Look at each case in turn.

1. Suppose n = 3k + 1

   Then $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$

   which is of the form 3t + 1 where $t = 3k^2 + 2k$

   But, by the assumption, n^2 is a multiple of 3 and this contradicts the fact that $n^2 = 3t + 1$

   The assumption is false and so n is not of the form 3k + 1

2. Suppose n = 3k + 2

   Then $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$

   which is of the form 3t + 1 for another integer t

   But, by the assumption, n^2 is a multiple of 3 and this contradicts the fact that $n^2 = 3t + 1$

   The assumption is false and so n is not of the form 3k + 2

The conjecture is therefore true and for any integer n, if n^2 is a multiple of 3 then n itself is a multiple of 3

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Irrationality is another common area for proof by contradiction and leads to a famous theorem which states that $\sqrt{2}$ is irrational. This is the subject of one of the questions in the exercise which is to come but the next example is very similar.

Example '$\sqrt{6}$ is not a rational number.' Prove this conjecture.

Answer:

Assume that $\sqrt{6}$ is rational.

Therefore $\sqrt{6} = \frac{p}{q}$ where p and q are positive integers with no common factors. (Otherwise the fraction could be reduced.) (p and q are relatively prime or coprime.)

So $6q^2 = p^2 \Rightarrow p^2$ is even.

Now if p is odd, then p^2 is odd.

But here p^2 is even: so **p is even**

and $p = 2m$ for some integer m

Thus $6q^2 = 4m^2 \Rightarrow 3q^2 = 2m^2$

Using the same argument again

$3q^2 = 2m^2 \Rightarrow q^2$ is even and so **q is even**.

Thus both p and q are even but both were assumed to have no common factors. This is a contradiction and the assumption that $\sqrt{6}$ is rational is false.

The original conjecture that $\sqrt{6}$ is not a rational number is therefore true.

There are many other techniques used in proofs, including the use of upper and lower bounds, largest or smallest integer and so on. With practice, recognising how to proceed does become easier.

Contradiction exercise

There is a similar exercise on the web.

20 min

Q12: Prove that $\sqrt{2} + \sqrt{3} < \sqrt{10}$ by using proof by contradiction.

Q13: If $m^2 = 14$ then m is not a rational number. Prove this conjecture using proof by contradiction.

Q14: If $x, y \in \mathbb{R}$ such that $x + y$ is irrational then at least one of x, y is irrational. Prove this using proof by contradiction.

Q15: Prove, by contradiction, that for any integer n, if n^2 is a multiple of 5 then n itself is a multiple of 5

Q16: 'There is no largest positive integer.' Prove this conjecture by contradiction.

Q17: Prove, using proof by contradiction, the Archimedean Property which states that given any two positive real numbers x and y, there is a positive integer n, such that $nx \geq y$

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Q18: Prove the conjecture that $\sqrt{2}$ is not a rational number using proof by contradiction.

Q19: 'There are infinitely many primes'. Prove this using proof by contradiction.

5.4 Further proof by induction

Learning Objective

Apply the method of proof by induction

Axioms are statements that are accepted without proof and the Peano axioms are a famous set which were constructed for the natural numbers in 1899.

The full set of axioms are shown in the section on divisibility rules and integer forms. One axiom of particular interest in this section is the axiom of induction that is the basis for the method of proof by induction. This states:

' If S is a subset of the natural numbers $\mathbb{N}$ such that

- if $1 \in S$
- if $x \in S$ then $x + 1 \in S$

then $S = \mathbb{N}$ '.

'Proof by induction (or proof by the principle of mathematical induction) is an advanced technique for showing that certain mathematical statements about the positive integers are true.

The proof has two parts: the *basis* and the *inductive step*.

The basis is an explicit check of the result for some positive integer.

The inductive step is as follows:

- Assume the statement to be true for some unspecified value of n, say n = k
- Use this to construct an argument that the statement is true for n = k + 1
- Once this is done the conclusion is that the statement is true for all positive integers $\geq$ the value of the basis step.'

The proofs of the Binomial Theorem and De Moivre's Theorem were given in topic 10 and in the relevant earlier topics 1 and 8. They will not be given again but it is important that both proofs are fully understood.

The technique is shown in the variety of examples which follow. Some may be familiar. Again it is possible to categorise the proofs and the first example shows one of the many summation results that can be proved using induction.

Example If n is a positive integer, then $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Prove the conjecture by using the principle of mathematical induction.

Answer:

Check the first value for which the conjecture is stated: when n = 1

then $1 = \frac{1}{2} \times 1\,(1 + 1) = 1$

Suppose the result is true for n = k then $1 + 2 + 3 + ... + k = \frac{1}{2}k\,(k + 1)$

Consider n = k + 1 then

$$1 + 2 + 3 + ... + k + k + 1 = \frac{1}{2}k\,(k + 1) + k + 1$$
$$= \frac{(k + 1)\,(k + 2)}{2}$$

The result is true for n = k + 1 if it is true for n = k. But it is true for n = 1 and so by the principle of mathematical induction, the conjecture is true.

It is always important to check for a starting value which relates to the conjecture and not assume that this will include n = 1. In fact many conjectures do not hold for smaller values. The next example shows an induction proof which has a basis step of n = 3 (and a conjecture which is untrue where n = 1 or 2).

Example $n^2 > 2n + 1$ when n is a positive integer ≥ 3. Prove this conjecture by the principle of mathematical induction.

Answer:

Check the first value of n = 3

Then $9 > 7$ and the result is true.

Assume that the result is true for n = k

Then $k^2 > 2k + 1$

Consider n = k + 1

Then $(k + 1)^2 = k^2 + 2k + 1 > 2k + 1 + 2k + 1 > 2k + 1 + 1 + 1 = 2\,(k + 1) + 1$

So $(k + 1)^2 > 2\,(k + 1) + 1$

The statement is true for n = k + 1 if it is true for n = k but it is true for n = 3. So by the principle of mathematical induction the conjecture is true for all n greater than or equal to 3.

The starting value for the conjecture is sometimes an essential link in the argument under the step for n = k + 1. This can be seen clearly in the next example where the fact that the conjecture is only being proved for $n \geq 3$ is used to reach a conclusion that $2 + 2n > 3$

Example $2^n > 1 + 2n$ for all $n \geq 3$. Prove this conjecture by using induction.

Check the first value for which the conjecture has to hold i.e. n = 3

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Then $2^3 > 1 + 6$ which is true.

Assume that the conjecture is true for n = k

Then $2^k > 1 + 2k$

Consider n = k + 1 (This has the effect of multiplying by 2 on LHS so multiply RHS by 2 also.)

So $2^{k+1} > 2(1 + 2k) = 2 + 2k + 2k$

(The next line is very important.)

But $2 + 2k + 2k > 3 + 2k$ since $2 + 2n > 3$ for all $n \geq 3$

and $3 + 2k = 1 + 2(k + 1)$

So $2^{k+1} > 1 + 2(k + 1)$

The result holds for n = k + 1 when it is true for n = k. But it is also true for n = 3 so by the principle of mathematical induction the conjecture is true.

Divisibility rules are also used.

Example Prove by induction that $3^{2n} + 7$ is divisible by 8 for $n \in \mathbb{Z}$

Answer:

Check the conjecture for n = 1

Then $3^2 + 7 = 16$ which is divisible by 8

Assume that the conjecture is true for n = k

Then $3^{2k} + 7 = 8a$ for some integer a

Consider n = k + 1

Then $3^{2(k+1)} + 7 = 3^2(3^{2k}) + 7 = 3^2(8a - 7) + 7$

$= 72a - 63 + 7 = 8(9a - 7)$

The result holds for n = k + 1 when it is true for n = k. But it is also true for n = 1 so by the principle of mathematical induction the conjecture is true.

Induction exercise

20 min

There is a web exercise for you to try if you wish.

Q20: By induction, if $n \in \mathbb{N}$, prove $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Q21: Prove by induction that $n^2 - 2n - 1 > 0$ for $n \geq 3$

Q22: Find the smallest integer m for which $n! > 3^n$
Hence prove by induction that $n! > 3^n$ for all $n > m$

Q23: Prove, by induction, that $n^2 - 1$ is divisible by 8 for all positive odd integers.

Q24: Prove by induction that $10^n \geq n^{10}$ for all $n \geq 10$

The induction principle is similar to the falling domino chain. If the dominoes are arranged correctly, knocking the first one over produces a chain reaction causing the rest to follow. Proof by induction is a very strong argument which works by the same principle. Care must be taken, however, to ensure that the correct starting point is chosen.

Activity

A famous number theorist, Fermat, conjectured that numbers of the form $2^{2^n} + 1$ are prime for all $\mathbb{N}$.

Inspect the first four numbers of this form and comment on the results. Is the fifth number prime? Use a graphics calculator to help by checking division by 641

This is a case where the initial results would seem to support his claim but even induction will fail to prove the result.

5.5 Proof using the contrapositive

Learning Objective

Use proof by assumimg the contrapositive appropriately

There are two types of **indirect** proof. The first is proof by contradiction which has been fully explored in this topic and in topic 10. The second is the subject of this section and is proof using the contrapositive.

Recall that earlier in the topic, the contrapositive implication of $P \Rightarrow Q$ was defined as $\neg Q \Rightarrow \neg P$. Although the two implications are logically equivalent, there are cases where it is easier to construct a proof starting with the contrapositive implication and facts about $\neg P$ and $\neg Q$.

Examples

1. The Pythagorean conjecture '$\sqrt{2}$ is irrational' is sometimes worded as 'if $x^2 = 2$, then x is not rational.' It is difficult to find properties and techniques for the two propositions '$x^2 = 2$' and 'x is not rational'. The contrapositive implication however reads 'if x is rational then $x^2 \neq 2$'.

2. Prove that if 7 is not a factor of n^2 then 7 is not a factor of n

Answer:

The conjecture looks cumbersome and is certainly a case for using the contrapositive. This states ' if 7 is a factor of n then 7 is a factor of n^2'

Thus $n = 7m$ for some $m \Rightarrow n^2 = 49m^2 = 7(7m^2)$

That is, 7 is a factor of n^2 if 7 is a factor of n. The conjecture is proved.

Proof using the contrapositive should only be used where it is easier to find relationships and properties for the negations than for the original propositions.

Contrapositive exercise

20 min

There is a web exercise for you to try if you prefer it.

Q25: Prove that if n^3 is not even, then n is not even using the contrapositive.

Q26: Prove by using the contrapositive that the integer n cannot be expressed as a sum of two square integers if n has remainder 3 on division by 4

Q27: Prove that if n^2 has the form 4m + 1 then n is odd.

Extra Help: Proofs - using the contrapositive

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

5.6 Direct methods of proof

Learning Objective

Apply direct methods of proof where possible

Direct methods of proof are the final type of proof to be examined.

There are no set rules for such a proof. It is constructed by using the properties and known results which apply to the propositions and the implication as its stands.

Saying this does not mean that the proof should be any less rigorous than the other methods of proof already used. In all cases, the deductions from one line of the proof to the next must be logical and correct mathematically.

The following example shows how a proof can fail by not applying logical and mathematically correct deductions.

This is an *incorrect* proof that 1 = 2

Let x = 1 and x = y

Then $xy = y^2$

So $xy - x^2 = y^2 - x^2$

$x(y - x) = (y + x)(y - x)$

Thus x = y + x and 1 = 2

This proof fails because the second last step divides through by y - x which of course is zero. As everyone should know, zero cannot be used as a divisor.

Now some correct proofs will be demonstrated. The first is very straightforward and uses the property that an integer greater than one can be expressed as m + 1 for some positive integer m.

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Example Prove that if $x > 1$ then $x^2 > 1$

Answer:

Let $x = m + 1$ where $m \geq 1$ then $x^2 = (m + 1)^2 = m^2 + 2m + 1$ and this is greater than 1 since $m \geq 1$

Therefore if $x > 1$ then $x^2 > 1$

As in other methods of proof, the divisibility rules and basic properties of numbers are used in this next example.

Example Prove that a number under 100 is divisible by 3 if the sum of its digits is divisible by 3.

Answer:

Let n be the number and $n = 10a + b$ where a and b are natural numbers ≤ 9

But $a + b = 3k$ by the conjecture that the sum of the digits is divisible by 3

So $n = 10a + b = 9a + a + b = 9a + 3k = 3(3a + k)$

The conjecture is proved.

The next example is straightforward algebraic manipulation.

Example Prove that $a \neq b$ if $a^2 + b^2 > 2ab$ for $a, b \in \mathbb{Z}$

Answer:

$a^2 + b^2 > 2ab$

$\Rightarrow a - 2ab + b^2 > 0$

$\Rightarrow (a - b)^2 > 0$

$\Rightarrow a - b > 0$ or $a - b < 0$

$\Rightarrow a > b$ or $a < b$

Thus $a \neq b$ as required.

In the next case the form of an odd integer is encountered again.

Example Prove that the square of an odd integer is odd.

Answer:

Let $n = 2m + 1$ for some integer m

Then $n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ which is in the form of an odd integer as required.

Summation results can sometimes be proved by this method and do not need induction.

Example Prove that for the arithmetic series with first term a and common difference d,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

Answer:

The solution uses a trick of expressing S_n in two different ways

$$S_n = \sum_{r=1}^{n}(a + (n - 1)d)$$
$$= a + (a + d) + (a + 2d) + ... + (a + (n - 2)d) + (a + (n - 1)d)$$

but

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + ... + (a + 2d) + (a + d) + a$$

Add the two expressions for S_n to give

$2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + ... + (2a + (n - 1)d) + (2a + (n - 1)d)$

That is $2S_n = n(2a + (n - 1)d)$

Hence $S_n = \frac{n}{2}[2a + (n - 1)d]$

There are also many summation results that can be proved using the common summations and the combination rules as in the next example.

Example Prove that $\sum_{r=1}^{n} r(r + 1) = \frac{1}{3}n(n + 1)(n + 2)$ by using the summation results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$

Answer:

By the combination rules and then using the summation results

$$\sum_{r=1}^{n} r(r + 1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$
$$= \frac{1}{6}n(n + 1)(2n + 1) + \frac{1}{2}n(n + 1)$$
$$= \frac{n(n + 1)(2n + 1) + 3n(n + 1)}{6}$$
$$= \frac{n(n + 1)(2n + 4)}{6}$$
$$= \frac{n(n + 1)(n + 2)}{3}$$

Direct proof exercise

20 min

There is a web exercise for you to try if you prefer it.

Q28: Prove that for all θ, $\cos^2\theta + \sin^2\theta = 1$

Q29: Prove that $\sum_{r=1}^{n}(2r - 1) = n^2$

Q30: Prove that if $a > 0$ and $b > c$ then $ab > ac$

Q31: Prove directly using the results of the summations to n terms of r, r^2 and r^3 that

$\sum_{r=1}^{n} r(r + 1)(r + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}$

Extra Help: Direct proof

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

5.7 The division algorithm and number bases

5.7.1 The division algorithm

Learning Objective

Use the division algorithm and produce the proof

At this point it is convenient to introduce an extra piece of symbolism. Consider the equation $6 = 3 \times 2$. A less precise way of stating this could be 2 divides 6 or 3 divides 6. For ease this is commonly written as $2 \mid 6$ or $3 \mid 6$.

This symbolism not only is a neat shorthand way of expressing division but also has another advantage over the words: when written in this form, $a \mid b$, then a is assumed $\neq 0$ provided that $b \neq 0$. If, in fact, a does not divide b this can be shown as $\nmid$

Obviously in the integers, $3 \nmid 7$ but to cope with this most people are taught at an early age to write this sum with a remainder such that $7 = 2 \times 3 + 1$

In general terms if there are two integers a and b $(a > b)$ where $b \nmid a$ then this can written as $a = qb + r$ where q and r are also integers. It is this type of division which forms the basis of the division algorithm.

There are two forms of the algorithm which differ only in the condition placed on the remainder r.

The original form of the division algorithm has the condition $0 \leq r < b$ where b is the divisor.

The general form of the division algorithm replaces this condition with $0 \leq r < \mid b \mid$ and is given in the definition which follows.

Division algorithm

If a and b are integers, $b \neq 0$, then there are unique integers q and r such that
$a = qb + r$ where $0 \leq r < \mid b \mid$

As in any division, q is the quotient and r is the remainder.

Note that the proof of the division algorithm is based on the 'the well ordering principle' which states that every nonempty set S of nonnegative integers contains a least element. The proof is shown for the original form with $0 \leq r < b$

Proof of the division algorithm

Let S be the set $\{a - xb$ such that $x \in \mathbb{Z}$ and $a - xb > 0\}$

The steps in the proof are:

- show that the set S is nonempty.
- show that S contains a least element.
- show that $r < b$
- show the uniqueness of q and r

- S is nonempty: by the condition on b in the definition of the division algorithm
  $b \geq 1$ so $|a| b \geq |a|$
  Thus $a - (-|a|)b = a + |a| b \geq a + |a| \geq 0$
  and so if $x = -|a|$, $a - xb \in S$
- S contains a least element: by the well ordering principle, since S is nonempty, it contains a least element say r such that $r = a - qb$ for some integer q and $r \geq 0$
- $r < b$: assume $r \geq b$ then
  $a - (q + 1)b = a - qb - b = r - b \geq 0$
  but $r - b < r$ which contradicts the fact that r is chosen as the least element in S. The original assumption is incorrect and $r < b$
- q and r are unique: suppose $a = bq_1 + r_1$ and $a = bq_2 + r_2$ with the conditions
  $0 \leq r_1 < b$ and $0 \leq r_2 < b$
  So $bq_1 + r_1 = bq_2 + r_2$
  $b(q_1 - q_2) = r_2 - r_1$
  But then the left hand side is a multiple of b and the condition on r_1 and r_2 dictates that
  $-b < r_2 - r_1 < b$
  But the only multiple of b between -b and b is zero.
  Hence $r_2 - r_1 = 0 \Rightarrow r_2 = r_1$ and $b(q_1 - q_2) = 0 \Rightarrow q_1 = q_2$

Division algorithm example

5 min

There is an example on the web of this algorithm.

Here are two examples of the division algorithm.

Examples

1. Find q and r by the division algorithm, such that $a = qb + r$
where $a = -29$ and $b = 6$

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Answer:

-29 = -5 . 6 + 1 where a = -29, q = -5, b = 6 and r = 1

2. Find q and r by the division algorithm, such that a = qb + r

where a = 33 and b = 4

Answer:

33 = 8 . 4 + 1 where a = 33, q = 8, b = 4 and r = 1

Note the convention of expressing the calculation using a dot. This is an important and useful approach to adopt with both the division and the Euclidean algorithm.

At first glance, the subtleties of the division algorithm may go unnoticed.

In example one note that, in keeping with the division algorithm r is positive.

So -29 = -4 . 6 - 5 is not a solution.

In example two, 33 = 7 . 4 + 5 is not a solution in keeping with the division algorithm since r $>$ b (5 $>$ 4).

Division algorithm exercise

5 min

There is another exercise on the web for you to try if you wish.

Q32: Using the division algorithm find q and r such that a = qb + r where

1. a = -42 and b = 5
2. a = 25 and b = -7
3. a = 14 and b = -9

Q33: Using the division algorithm find q and r such that a = qb + r where

1. a = -1 and b = 3
2. a = 4 and b = 9
3. a = -12 and b = 3

It may not have been apparent at the time but the division algorithm has been used in some form in many of the proofs in the previous sections. The application of the algorithm produces the integer forms such as 2k for an even integer and 2k + 1 for an odd integer.

It is the condition $0 \leq r < | b |$ that is used to great effect in determining the form that a particular number can take.

Example Suppose that b = 4 then in the equation a = qb + r, the division algorithm shows that a will have one of the following forms

- 4k where r = 0 with q = k
- 4k + 1 where r = 1
- 4k + 2 where r = 2

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- 4k + 3 where r = 3

In all cases r satisifes the condition $0 \leq r < |\ b\ |$

Q34: Determine all the even forms of a number on division by 8

These types of integer forms are the basis of many proofs.

5.7.2 Number bases

Learning Objective

Convert decimal integers into other number bases such as binary, octal or hexadecimal

The division algorithm is a simple method to use to convert integers to different number bases. The standard number base used in mathematics is 10, that is the place values of a number are powers of 10. The decimal system, as it is known, has the structure as shown.

The number 25, 367

$$\begin{array}{ccccc} \mathbf{10^4} & \mathbf{10^3} & \mathbf{10^2} & \mathbf{10^1} & \mathbf{10^0} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 5 & 3 & 6 & 7 \end{array}$$

2 x 1000 + 5 x 1000 + 3 x 100 + 6 x 10 + 7 x 1

A number can be converted and expressed in a different number system using any integer as a number base, but three of the most useful are the bases 2, 8 and 16. These are called binary, octal and hexadecimal respectively and are used extensively in subjects such as computing and electronics.

Binary

In the decimal number system there are ten digits, 0 to 9. In binary, which has a base of two, there are just two digits, 0 and 1. The place values in a binary system are all powers of 2 and another diagram showing these place values can be constructed.

The binary number 11010

$$\begin{array}{ccccc} \mathbf{2^4} & \mathbf{2^3} & \mathbf{2^2} & \mathbf{2^1} & \mathbf{2^0} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 \end{array}$$

The diagram shows how easy it is to find the decimal form of a binary number.

11010 is 1 x 2^4 + 1 x 2^3 + 1 x 2^1 = 26

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The division algorithm is used to find the binary form from the decimal form and is best described by an example.

Example Convert 30 into a binary number.

Answer:

Use the algorithm repeatedly for a = qb + r; in the first case a = 30 and b = 2 (for binary); then a = 15 and b = 2 and so on until the algorithm can be taken no further.

30 = 15 . 2 + 0

15 = 7 . 2 + 1

7 = 3 . 2 + 1

3 = 1 . 2 + 1

1 = 0 . 2 + 1

The binary number is taken directly from the remainders in this calculation by reading them off *upwards* and is 11110

Octal

This is just as straightforward but uses the powers of 8 as place values. The octal system only uses the eight integers 0 to 7

Example Convert the number 53467 into octal form.

Answer:

In this case the equation a = qb + r has a = 53467 and b = 8

53467 = 6683 . 8 + 3

6683 = 835 . 8 + 3

835 = 104 . 8 + 3

104 = 13 . 8 + 0

13 = 1 . 8 + 5

1 = 0 . 8 + 1

Again the octal number is taken straight from the remainders in the calculation by reading upwards and is 150333

This can be checked in reverse as 3 x 1 + 3 x 8 + 3 x 8^2 + 5 x 8^4 +1 x 8^5 = 53467

Hexadecimal

This system uses base 16 and has sixteen 'digits'. Since each digit can only occupy one place value a slight modification is needed. In this case the system uses integers 0 to 9 and then A for 10, B for 11, C for 12, D for 13, E for 14 and F for 15

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Example Find the hexadecimal form of the number 298047

Answer:

Repeated application of the division algorithm with b = 16 gives

298047 = 18627 . 16 + 15 note: 15 $\equiv$ F

18627 = 1164 . 16 + 3

1164 = 72 . 16 + 12 note: 12 $\equiv$ C

72 = 4 . 16 + 8

4 = 0 . 16 + 4

The hexadecimal number read upwards is 48C3F

Number bases exercise

10 min

There is a web exercise for you to try if you prefer.

Q35: Convert the following numbers into binary numbers:

a) 332
b) 501
c) 43
d) 63

Q36: Find the octal forms of the following numbers:

a) 347
b) 2924
c) 3012
d) 534

Q37: Find the hexadecimal equivalent numbers to

a) 4014
b) 364
c) 2179
d) 5034

’Mathematics is the queen of sciences and number theory is the queen of mathematics.’

Carl Gauss 1777 - 1855

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5.8 The Euclidean algorithm and its applications

In the last section the definition of the division algorithm gives expressions of the form a = qb + r

Consider what happens when r = 0

In this case, the division algorithm leads to a definition. 'An integer b is said to be divisible by a, that is a|b if b = ac for some integer c. So a is a divisor of b.'

Suppose that there is an integer d such that d | a and d | b for two integers a and b, one non-zero. In this case d is a common divisor of a and b. An extra condition gives a definition for the greatest common divisor.

Greatest common divisor

The greatest common divisor of two integers a and b, where at least one is nonzero, denoted gcd (a, b) is the positive integer d such that:

- d | a and d | b
- if c | a and c | b then c $\geq$ d

The greatest common divisor is also well known as the highest common factor (hcf).

Example Find gcd (42, 68)

Answer:

The positive divisors of 42 are 1, 2, 3, 6, 7, 14, 21, 42 and for 78 they are 1, 2, 3, 6, 13, 26, 39, 78. The common divisors are 1, 2, 3, 6 of which 6 is the greatest.

gcd (42, 68) = 6

A particular case occurs when, for two integers a and b, the gcd (a, b) = 1

Relatively prime

The integers a and b, where at least one is non-zero, are relatively prime (coprime) when gcd (a, b) = 1

Q38: Show that 2210 and 399 are relatively prime.

It would be tedious to take large integers and search for their factors, as the last question demonstrates. In fact the division algorithm again plays its part in producing another famous algorithm to make this task much easier.

5.8.1 The Euclidean algorithm

Learning Objective

Use the Euclidean algorithm to find the greatest common divisor of two integers

This algorithm is very similar to the calculation required for the change of number basis except that at each step the value of b also changes. By repeated use of the division algorithm the calculation is subsequently reduced to give the greatest common divisor in the final step. An example will help to demonstrate this.

Example Find the gcd of 140 and 252

Answer:

Using the expression $a_1 = qb_1 + r_1$ let a be the larger number

$a_1 = 252$ and then $b_1 = 140$

Use the division algorithm to obtain

252 = 1 . 140 + 112

At this stage a_2 becomes 140 and b_2 becomes 112 and the next step gives

140 = 1 . 112 + 28

The process continues

112 = 4 . 28 + 0

Once r = 0 after i steps the algorithm stops and the last value of b_i gives the greatest common divisor of the two numbers.

Here is diagram indicating the steps in the algorithm for the last example.

To find the gcd of 252 and 140 with the Euclidean algorithm

a = q . b + r

252 = 1 . 140 + 112

140 = 1 . 112 + 28

112 = 4 . 28 + 0

STOP

Euclidean algorithm

10 min

On the web there is a demonstration of the algorithm which allows you to input two integers and watch the algorithm in action.

It is common to write the gcd of two numbers as, for example, gcd (252, 140) = 28 as in the last diagram.

Example Find the gcd of 1980 and 3696

Answer:

Let the larger number be a so a = 3696 and let b = 1980. Now use the division algorithm to give

3696 = 1 . 1980 + 1716

1980 = 1 . 1716 + 264

1716 = 6 . 264 + 132

264 = 2 . 132 + 0

The greatest common divisor (or highest common factor) of 3696 and 1980 is 132

An interesting use of this algorithm is to reduce fractions.

Example Simplify the fraction $\frac{19712}{55040}$

Answer:

Use the Euclidean algorithm to find the gcd.

55040 = 2 . 19712 + 15616

19712 = 1 . 15616 + 4096

15616 = 3 . 4096 + 3328

4096 = 1 . 3328 + 768

3328 = 4 . 768 + 256

768 = 3 . 256 + 0

Thus the gcd is 256 and now divide both numerator and denominator by 256 to give the fully simplified fraction as $\frac{77}{215}$

Euclidean algorithm exercise

15 min

There is a web version of this exercise for you to try if you prefer it.

Q39: Use the Euclidean algorithm to find the gcd of 299 and 1365

Q40: What is the highest common factor of 5187 and 760

Q41: Use the Euclidean algorithm to show that 10465 and 5643 are coprime.

Q42: Use the Euclidean algorithm to find the gcd of 3024 and 5184

Q43: Reduce the fraction $\frac{372}{4340}$ to its simplest form.

Activity

It is possible, in a few simple steps, to program a graphics calculator to do this algorithm and give the gcd. Try this if you have time.

Fibonacci numbers

The Fibonacci sequence, mentioned in topic 9 on sequences and series, produces some fascinating results in this topic.

Recall that the sequence is defined by $u_{n+2} = u_{n+1} + u_n$ where $u_1 = 1$ and $u_2 = 1$

The sequence is: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

It can be shown that gcd $(u_n, u_{n+1}) = 1$ for all $n \geq 1$. So successive Fibonacci numbers are relatively prime.

Activity

Try to prove this result, viz, gcd $(u_n, u_{n+1}) = 1$ for all $n \geq 1$

Now consider non-successive Fibonacci numbers by first answering the following question.

Q44: Find gcd (610, 8) and gcd (610, 55)

The last question used the Fibonacci numbers 8, 55 and 610. The answers were also the Fibonacci numbers 2 and 5 and in general it can be proved that the greatest common divisor of two Fibonacci numbers is in fact another Fibonacci number. This result can be taken further.

The question first asked for gcd (610, 8). These are the Fibonacci numbers u_{15} and u_6 and the answer of a greatest common divisor of 2 is the Fibonacci number u_3

Thus the gcd $(u_{15}, u_6) = u_3$ and this leads to a very useful theorem.

The greatest common divisor of two Fibonacci numbers is also a Fibonacci number such that

gcd $(u_n, u_m) = u_d$ where gcd $(n, m) = d$

Example Find the gcd (987, 144)

Answer: $987 = u_{16}$ and $144 = u_{12}$

But gcd (16, 12) = 4 and so gcd (987, 144) = $u_4 = 3$

Activity

Use the Euclidean algorithm and check that the gcd (987, 144) = 4

Activity

Investigate the proposition that every positive integer can be expressed as a sum of Fibonacci numbers with none used more than once. Try to prove it as a challenge.

5.8.2 The gcd as a linear combination

Learning Objective

Use the Euclidean algorithm to express the greatest common divisor as a linear combination of the two integers a and b

Having used the Euclidean algorithm to find the gcd of two integers it is possible to take the algorithm and retrace the steps taken to arrive at a linear combination of the two integers which equals the gcd.

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Example Use the Euclidean algorithm to obtain integers x and y which satisfy the equation

gcd (2695, 1260) = 2695x + 1260y

Answer:

First use the Euclidean algorithm to find the gcd

2695 = 2 . 1260 + 175

1260 = 7 . 175 + 35

175 = 5 . 35 + 0

The gcd (2695, 1260) = 35

Now taking the algorithm it is possible using algebraic manipulation to work *backwards*. Start at the second last step of the algorithm and work back each step eliminating the remainders.

working down	working up
	35 = -7 . 2695 + 15 . 1260
2695 = 2 . 1260 + 175	35 = 1260 - 7(2695 - 2 . 1260)
1260 = 7 . 175 + 35	35 = 1260 - 7 . 175
175 = 5 . 35 + 0	

So 35 = -7 . 2695 + 15 . 1260 and x = -7 and y = 15

Thus the gcd is found in the last entry in the first column and the linear combination, having worked upwards is given in the first entry of the second column.

Note that this is not the only way to express the gcd as a linear combination but it will always give a solution. The set of all solutions can, however, be found from this solution. A simpler example will be easier to follow.

Example Find all solutions of the equation gcd (48, 20) = 48x + 20y

Answer:

Using the Euclidean algorithm gives:

48 = 2 . 20 + 8

20 = 2 . 8 + 4

8 = 2 . 4 + 0

Working backwards to find one linear combination gives

4 = 20 - 2 . 8

4 = 20 - 2(48 - 2 . 20) = 5 . 20 - 2 . 48

Thus one solution is 4 = 48x + 20y where x = -2 and y = 5

So 48x + 20y = -2 . 48 + 5 . 20

Rearranging this gives

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48(x + 2) = 20(5 - y)Now divide through by the gcd which is 4 to give

12(x + 2) = 5(5 - y)

Since 12 and 5 have no common factors x + 2 must be a multiple of 5 and 5 - y is a multiple of 12

Thus x + 2 = 5m $\Rightarrow$ x = 5m - 2

5 - y = 12m $\Rightarrow$ y = 5 - 12m

The general solution is 4 = 48(5m - 2) + 20(5 - 12m)

There is a shortcut to the general solution.

Suppose that ax + by = d where d is the gcd (a, b)

If, by the Euclidean algorithm, the particular solutions for x and y are x_0 and y_0 then the general solutions x_n and y_n are found as follows

$x_n = x_0 + \left(\frac{b}{d}\right) m$

and

$y_n = y_0 - \left(\frac{a}{d}\right) m$

In the last example the particular solutions were $x_0 = -2$ and $y_0 = 5$

The coefficient of x = b = 48 and the coefficient of y = a = 20

The gcd is 4

Thus the general solution is $4 = 48x_n + 20y_n$

$4 = 48(-2 + ({}^{20}/_{4})m) + 20(5 - ({}^{48}/_{4})m)$

That is, 4 = 48(5m - 2) + 20(5 - 12m)

A linear combination of two integers equal to the gcd is a special example of the simplest type of Diophantine equation.

Diophantine equation

A Diophantine equation is an equation with integer coefficients in one or more unknowns which is to be solved in the integers.

In its simplest form in two unknowns the linear Diophantine equation is ax + by = c

If the gcd (a, b) = d is found then such an equation has a solution if and only if $d \mid c$

The full method of then solving the equation is very similar to that for finding the general solution of the equation ax + by = c; with one extra step. Suppose that c = dn then after finding the particular solution ax + by = d the whole equation is multiplied up to give nax + nby = dn = c and the general solution is found in the same way as before.

Example Find the solution to the linear Diophantine equation 18 = 90x + 12y if it exists.

Answer:

First find the gcd (90, 12)

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$90 = 7 \cdot 12 + 6$

$12 = 2 \cdot 6 + 0$

The gcd (90, 12) = 6

Since $6 \mid 18$ there is a solution to the Diophantine equation.

Working backwards $6 = 90 \cdot 1 + 12 \cdot (-7)$

Multiply by 3 to give $18 = 90 \cdot 3 + 12 \cdot (-21)$ which is one particular solution to the Diophantine equation.

Using the shortcut the general solution is

$18 = 90(3 + (^{12}/_{6})m) + 12(-21 - (^{90}/_{6})m) = 90(3 + 2m) + 12(-21 - 15m)$

Note: Fermat's conjecture or, as more commonly known, Fermat's last theorem states that there is no solution to the Diophantine equation $x^n + y^n = z^n$ in the integers for $n > 2$. This was one of the most famous 'unproved' results until Andrew Wiles presented a proof in the mid 1990s.

Linear combinations exercise

There is another exercise on the web to find linear combinations of the gcd of two integers and to solve linear Diophantine equations.

20 min

Q45: Use the Euclidean algorithm to find integers x and y such that

gcd (693, 84) = 693x + 84y

Q46: Use the Euclidean algorithm to find integers x and y

such that gcd (10080, 3705) = 10080x + 3705y

Q47: Use the Euclidean algorithm to find a solution to the equation

gcd (336, 180) = 336x + 180y

Q48: Use the Euclidean algorithm to solve for x and y when 14 = 1078x + 420y

Q49: Find the general solution of the equation gcd (585, 104) = 585x + 104y using the Euclidean algorithm.

Q50: Solve the linear Diophantine equation 20 = 204x + 56y

5.9 Summary

At this stage you should be familiar with the following techniques and the methods:

- Use of the correct implication.
- Proof by contradiction.
- Proof by induction.

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- Proof using the contrapositive.
- Direct methods of proof.
- The division algorithm.
- The Euclidean algorithm.
- Uses of the Euclidean algorithm.

5.10 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web which give a selection of interesting sites to visit. These sites can lead to an advanced study of the topic but there are many areas which will be of passing interest.

Aristotle

His logical arguments are the basis of much of the mathematical logic of the present day.

Euclid

The Euclidean algorithm is given in Euclid's seventh book of the Elements but there is evidence to suggest that it was known before this time.

de Morgan

An 18th century mathematician who formalised the method of proof by induction. The method however was known long before.

Boole

This English mathematician showed in 1847 that algebra could be used to express logical relationships. His work referred to as Boolean algebra is widely used in electronics.

Peano

He was a relatively modern Italian mathematician and logician who lived in the late 19th/early 20th century. His axioms form a sound basis for the study of integers. He also introduced the membership sign $\in$ in his work on set theory.

Lame

The French mathematician Gabriel Lame produced a theorem which states that the number of steps required in the Euclidean algorithm is never greater than 5 times the number of digits in the lesser of the two numbers.

Godel

A 20th century mathematician famous for his incompleteness theorem. Look this one up on the web.

5.11 Divisibility rules and integer forms

The Peano axioms are stated below in their original form. Later Peano modified his axioms to include zero.

- 1 is a natural number.
- Every natural number x has a successor $x' = (x + 1)$
- There is no number whose successor is one. (no x such that $x + 1 = 1$)
- The function 'successor' is a one to one function. (if $x' = y'$ then $x = y$ and no two numbers have the same immediate successor)
- Induction: If S is a subset of the natural numbers $\mathbb{N}$ such that
  - 1 belongs to S
  - if x belongs to S then x' belongs to S

  then $S = \mathbb{N}$

Here the some of the divisibility rules for integers:

An integer N is divisible by:

2	if the last digit is even
3	if the sum of its digits is divisible by 3
4	if the number formed by the last two digits is divisible by 4
5	if the last digit is 5 or 0
6	if it is divisible by 2 and 3
7	if the number formed by removing the last digit and subsequently subtracting twice the last digit is divisible by 7
8	if the number formed by the last three digits is divisible by 8
9	if the sum of its digits is divisible by 9
10	if it ends in 0

The following integer forms are useful to know:

- An odd integer has the form $2m + 1$ for m an integer.
- An even integer has the form $2m$ for m an integer.
- A rational integer has the form $^p/_q$ for p and q integers.
- An integer is of the form $2k$ or $2k + 1$

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- An integer is of the form 3k, 3k + 1 or 3k + 2
- An integer is of the form 4k, 4k + 1, 4k + 2 or 4k + 3 and so on.

- Every odd integer is of the form 4k + 1 or 4k + 3
- The square of any integer is of the form 3k or 3k + 1
- The cube of any integer is of the form 9k, 9k + 1 or 9k + 8

- The product of two or more integers of the form 3k + 1 is of the same form.
- The product of two or more integers of the form 4k + 1 is of the same form.

5.12 Review exercise

Review exercise

15 min

There is an alternative exercise available on the web for you to try if you prefer.

Q51: Use proof by induction to prove that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

Q52: Use the Euclidean algorithm to find the greatest common divisor of the integers 943 and 779

Q53: Prove, by induction, that the sum of the cubes of any three successive integers is divisible by 9

Q54: Use the Euclidean algorithm to find the greatest common divisor of 5141 and 3763

5.13 Advanced review exercise

Advanced review exercise

15 min

There is an alternative exercise available on the web for you to try if you prefer.

Q55: Use the Euclidean algorithm to find integers x and y such that

247x + 139y = 1

Q56: Use the Euclidean algorithm to find integers x and y such that

252x + 160y = 4

Q57: Use the Euclidean algorithm to find integers x and y such that

gcd(297, 180) = 297x + 180y

5.14 Set review exercise

Set review exercise

20 min

The answers for this exercise are only available on the web by entering the answers obtained in an exercise called 'set review exercise'. The questions may be structured differently but will require the same answers. One question on proof and one on the Euclidean algorithm should be completed in 20 minutes. However, if time permits, try them all.

Q58: Use proof by induction to show that $\sum_{r = 2}^{n} r(r - 1) = \frac{n(n + 1)(n - 1)}{3}$

Q59: Use the Euclidean algorithm to find the greatest common divisor of 1081 and 2773

Q60: Use proof by induction to prove $\sum_{r = 2}^{n} 2r(r - 1)(r + 1) = \frac{n(n - 1)(n + 1)(n + 2)}{2}$

Q61: Use the Euclidean algorithm to find the greatest common divisor of 3053 and 1349

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Topic 6

End of unit three tests

Contents

Online tests are provided on the web. The first two tests are set to provide questions at level 'C' competency and cover work from the five topics for unit 3.

The third test gives questions at level 'A' or 'B'.

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Glossary

addition of two matrices

If A and B are two matrices of the same order, the matrix A + B is formed by adding the corresponding elements of each matrix.

auxiliary equation

The second order, linear, homogeneous differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

has **auxiliary equation**

$am^2 + bm + c = 0$

cartesian equation of a plane

The Cartesian equation of a plane containing the point P (a_1, a_2, a_3) and perpendicular to the vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ is $xn_1 + yn_2 + zn_3 = d$ where $d = a_1n_1 + a_2n_2 + a_3n_3$

column matrix

A column matrix has only one column but any number of rows.

complementary function

The non-homogeneous equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

has a corresponding homogeneous equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

The homogenous equation has a general solution, y_c, which contains two constants.

y_c is the **complementary function** for the non-homogeneous equation.

contrapositive

The contrapositive of the implication $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$

convergent series

A **convergent series** is one for which the limit of partial sums exists. This limit is called the sum and is denoted by S_∞

converse

The converse of the statement $P \Rightarrow Q$ is $Q \Rightarrow P$

determinant of a matrix

The determinant of a matrix is a value representing sums and products of a square matrix.

If the matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

then the determinant of A is $\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

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determinant value of a 2 x 2 matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the value of det $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

determinant value of a 3 x 3 matrix

If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then the value of det A =

$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} =$

$a(ei - hf) - b(di - gf) + c(dh - ge)$

dilatation or scaling matrix

The matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ acting by multiplication on the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ represents the effect on the point (x, y) thus scaling the x-coordinate by λ and the y-coordinate by μ

diophantine equation

A Diophantine equation is an equation with integer coefficients in one or more unknowns which is to be solved in the integers.

divergent series

A **divergent series** is one which is not convergent. For example 1 + 2 + 3 + 4 + 5 ... is a divergent series. The sum of this series will continue increasing the more terms we add. It will not tend towards a limit.

division algorithm

If a and b are integers, $b \neq 0$, then there are unique integers q and r such that $a = qb + r$ where $0 \leq r < |b|$

elementary row operations

The three ways in which a matrix can be manipulated to solve a system of equations are called elementary row operations. They are:

- Interchange two rows.
- Multiply one row by a non-zero constant.
- Change one row by adding a multiple of another row.

equation of simple harmonic motion

The equation

$\frac{d^2x}{dt^2} + w^2x = 0$

is the **equation of simple harmonic motion**.

It has general solution

$x = A \cos wt + B \sin wt$

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equivalent to

Is equivalent to means that the first statement implies and is implied by the second statement. (The first statement is true **if and only if** the second statement is true.)

first order linear differential equation

A **first order linear differential equation** is an equation that can be expressed in the standard form

${}^{dy}/_{dx} + P(x)y = f(x)$

first order recurrence relation

A **first order recurrence relation** is a recurrence relation where u_{n+1} depends only on u_n and not on values of u_r where $r < n$

fixed point

The recurrence relation $u_{n+1} = au_n + b$ has **fixed point** given by ${}^{b}//_{(1-a)}$. The recurrence relation will only converge to this fixed point if $(-1 < a < 1)$

general solution

The **general solution** of a differential equation contains one or more arbitrary constants and gives infinitely many solutions that all satisfy the differential equation.

greatest common divisor

The greatest common divisor of two integers a and b, where at least one is nonzero, denoted gcd (a, b) is the positive integer d such that:

- $d \mid a$ and $d \mid b$
- if $c \mid a$ and $c \mid b$ then $c \geq d$

homogeneous

A second order, linear, differential equation is **homogeneous** when it can be expressed in the standard form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

where a, b and c are constants and $a \neq 0$

identity matrix

An n x n matrix which has only ones on the leading diagonal and zeros elsewhere is called an identity matrix and takes the form

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & .. & 0 \\ 0 & 1 & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & 1 \end{pmatrix} \updownarrow n \\ \leftarrow n \rightarrow \end{array}$$

It is denoted by I.

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initial condition

For a differential equation an **initial condition** is an additional condition which must be satisfied by the solution. This could be a coordinate on a curve, a velocity at $t = 0$, the amount of money in a bank account on 1st January 2000, etc.

integrating factor

For the first order linear differential equation $\frac{dy}{dx} + P(x)y = f(x)$, the **integrating factor** is given by $I(x) = e^{\int P(x)\,dx}$

inverse

The inverse of the statement $P \Rightarrow Q$ is $\neg P \Rightarrow \neg Q$

inverse of a 2 x 2 matrix

Let A be the square non-singular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The inverse of A is denoted $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

iteration

Iteration is the successive repetition of a mathematical process using the result of one stage as the input for the next.

length of a vector

Let **p** be the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then the length of **p** is defined as $|\mathbf{p}| = \sqrt{a^2 + b^2 + c^2}$

linear first order recurrence relation

A **linear first order recurrence relation** is a recurrence relation of the form $u_{n+1} = au_n + b,\ a \neq 0$

maclaurin's theorem

Maclaurin's theorem states that

$$f(x) = \sum_{r=0}^{\infty} f^{(r)}(0)\frac{x^r}{r!}$$

$$= f(0) + f^{(1)}(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \ldots + f^{(n)}(0)\frac{x^n}{n!} + \ldots$$

matrix

A matrix is a rectangular array of numbers.

multiplication by a scalar

To multiply an m x n matrix A by a scalar λ, take each element and multiply it by λ to give the matrix λA.

multiplication of matrices

The product of the two matrices A_{mn} and $B_{np} = C_{mp}$

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If $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$

then the matrix AB =

$$\begin{pmatrix} a_{11}b_{11} + \dots + a_{1n}b_{n1} & a_{11}b_{12} + \dots + a_{1n}b_{n2} & \dots & a_{11}b_{1p} + \dots + a_{1n}b_{np} \\ a_{21}b_{11} + \dots + a_{2n}b_{n1} & a_{21}b_{12} + \dots + a_{2n}b_{n2} & \dots & a_{21}b_{1p} + \dots + a_{2n}b_{np} \\ \dots & \dots & \dots & \dots \\ a_{m1}b_{11} + \dots + a_{mn}b_{n1} & a_{m1}b_{12} + \dots + a_{mn}b_{n2} & \dots & a_{m1}b_{1p} + \dots + a_{mn}b_{np} \end{pmatrix}$$

necessary condition

If $P \Rightarrow Q$ then Q is a necessary condition for P.

negation

- If the proposition P is true then its negation $\neg P$ is false.
- If the proposition P is false then its negation $\neg P$ is true.

non-homogeneous

A second order, linear, differential equation is **non-homogeneous** when it can be expressed in the standard form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

where a, b and c are constants and $a \neq 0$ and $f(x) \neq 0$

numerical analysis

Numerical analysis is a branch of mathematics. It can be described as the analysis and solution of problems which require calculation.

order of convergence

For an iterative process in the form $x_{n+1} = g(x_n)$

The **order of convergence** is first order when $|g'(x)| < 1$ but $|g'(\alpha)| \neq 0$

The order of convergence is second order when $|g'(x)| < 1$ and $|g'(\alpha)| = 0$ but $|g''(\alpha)| \neq 0$

orthogonal matrix

A square matrix A is orthogonal $\Leftrightarrow A^T = A^{-1}$

$A^T = A^{-1} \Rightarrow \det A = \pm 1$

parametric equation of a plane

The parametric equation of the plane is

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ where **a** is a position vector of a point in the plane, **b** and **c** are two non-parallel vectors parallel to the plane and λ and μ are real numbers.

parametric form of the equation of a line

The parametric equations of a line through the point $P = (a_1, a_2, a_3)$ with direction $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ are

$x = a_1 + \lambda d_1$, $y = a_2 + \lambda d_2$, $z = a_3 + \lambda d_3$ where λ is a real number.

particular integral

y_p is a **particular integral** of the differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

when y_p is a solution to the equation.

particular solution

The **particular solution** of a differential equation is a solution which is often obtained from the general solution when values are assigned to the arbitrary constants.

position vector

A position vector is a vector which starts at the origin.

power series

A **power series** is an expression of the form

$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$

where $a_0, a_1, a_2, a_3, ..., a_n, ...$ are constants and x is a variable. It is called a power series as it is made up of a sequence of powers of x with coefficients $a_0, a_1, a_2, a_3, ..., a_n, ...$

proof by contradiction

Proof by contradiction assumes the conjecture to be false and shows that this leads to the deduction of a false conclusion.

recurrence relation

A **recurrence relation** describes a sequence of terms $u_0, u_1, u_2, u_3, ..., u_n, u_{n+1}$, ... where each term is a function of previous terms.

reflection matrix

The matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ acting, by multiplication on the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, represents the effect of reflection on the point (x, y) in the line through the origin which makes an angle of $\theta°$, $-90° \leq \theta \leq 90°$

relatively prime

The integers a and b, where at least one is non-zero, are relatively prime (coprime) when gcd (a, b) = 1

rotation matrix

The matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ acting, by multiplication on the column matrix $A = \begin{pmatrix} x \\ y \end{pmatrix}$, represents the effect of rotation through $\theta°$ anticlockwise about the origin on the point (x, y)

row matrix

A row matrix has only one row but any number of columns.

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scalar product in component form

If $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

then the scalar product is the number $\mathbf{p} \bullet \mathbf{q}$ = ad + be + cf

scalar product in geometric form

The scalar product of two vectors **a** and **b** is defined as

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ where θ is the angle between **a** and **b**, $0 \leq \theta \leq 180°$

scalar triple product

The scalar triple product **a • (b x c)** is the number given by the scalar product of the two vectors **a** and (**b x c**)

second order, linear, differential equation

A **second order, linear, differential equation** is an equation which can be expressed in the standard form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

where a, b and c are constants and $a \neq 0$.

second order recurrence relation

A **second order recurrence relation** is a recurrence relation of the form $u_{n+2} = au_{n+1} + bu_n + c$, with a and $b \neq 0$. Note that u_{n+2} depends on the previous two terms in the sequence.

singular matrix

A matrix whose determinant is zero is called a singular matrix.

square matrix

A square matrix has the same number of rows and columns.

subtraction of two matrices

If A and B are two matrices of the same order, the matrix A - B is formed by subtracting the corresponding elements of matrix B from those in matrix A.

sufficient condition

If $P \Rightarrow Q$ then P is a sufficient condition for Q.

symmetric form of the equation of a line

If $x = a_1 + \lambda d_1$, $y = a_2 + \lambda d_2$, $z = a_3 + \lambda d_3$ are parametric equations of a line, the symmetric equation of this line is

$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$

transpose of a matrix

The transpose of a matrix A of order $m \times n$ is found by reflecting the matrix in its main diagonal.

It is denoted A^T or A'.

The effect is to interchange the rows of the matrix with the columns and A^T has order $n \times m$.

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vector equation of a plane

The vector equation of a plane containing the point P and perpendicular to the vector **n** is **(r - a) •n** = 0 where **a** = $\overrightarrow{OP}$

vector form of the equation of a line

The vector equation of a line through the point P with direction **d** is

r = **a** + λ**d** where **a** = $\overrightarrow{OP}$, **d** is a vector parallel to the required line and λ is a real number.

vector product in component form

If **p** = a**i** + b**j** + c**k** and **q** = d**i** + e**j** + f**k**

then the **vector product** is defined as the vector

p x q = (bf - ec)**i** - (af - dc)**j** + (ae - db)**k**

vector product in geometric form

The vector product of **a** and **b** is defined with

- magnitude of $|$ **a x b** $|$ = $|$ **a** $||$ **b** $|$ sin θ, where θ is the angle between **a** and **b**, $0 \leq \theta \leq 180°$
- direction perpendicular to both vectors **a** and **b** as determined by the right hand rule.

vector quantity

A vector quantity is a quantity which has both direction and magnitude.

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Hints for activities

Topic 4: Further Ordinary Differential Equations

Save the fish

Hint 1: Let S = total mass of salt in the tank in and let w = the flow of water in litres/minute. Then we can write down the differential equation

$\frac{dS}{dt} = -\frac{w}{5000}S$

In standard linear form this becomes

$\frac{dS}{dt} + \frac{w}{5000}S = 0$

with $P(t) = \frac{w}{5000}$ and $f(t) = 0$

Hint 2:

We can now obtain the differential equation

$\frac{d}{dt}\left(e^{-wt/5000}S\right) = 0$

which we can integrate and solve for S to give

$S = Ce^{-wt/5000}$ where C is an arbitrary constant.

Hint 3: There is initially $S(0) = \frac{20 \times 5000}{1000} = 100$kg of salt in the tank.

Therefore C = 100.

We need there to be $S = \frac{12 \times 5000}{1000} = 60$kg when the fish is put in.

We can therefore write down the equation

$60 = 100e^{-wt/5000}$

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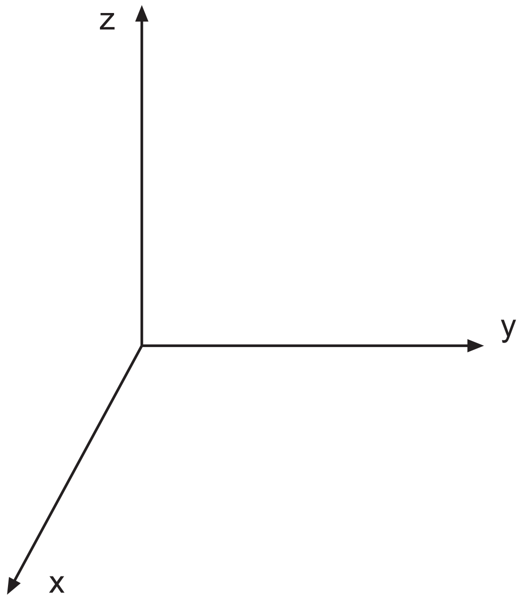
Answers to questions and activities

1 Vectors

Revision exercise (page 3)

Q1:



Q2: The equation will take the form y = mx + c

The gradient m = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{15}{5} = 3$

Thus y = 3x + c

By substituting the values for one of the points into this equation the value of c is obtained and is -7

The equation of the line is y = 3x - 7

Q3: $(ab + 2c)(ac - b)(c + b) = (a^2bc + 2ac^2 - ab^2 - 2bc)(c + b)$

$= a^2bc^2 + 2ac^3 - ab^2c - 2bc^2 + a^2b^2c + 2abc^2 - ab^3 - 2b^2c$

If $a^2 = 2$, $b^2 = 1$ and $c^2 = 1$ then this simplifies to a(b + c)

Q4: x = 1, y = -1 and z = 3

Vector length exercise (page 7)

Q5: The vector to the point (-1, -2, 4) has length 4.58 to 2 decimal places.

The vector to the point (2, 5, 4) has length 6.71 to 2 decimal places.

The vector to the point (4, 2, -5) has length 6.71 to 2 decimal places.

The vector to the point (3, -5, -2) has length 6.16 to 2 decimal places.

Q6: The length is $\sqrt{(-2)^2 + 2^2 + 1^2} = 3$

Vector notation exercise (page 7)

Q7: -3**i** + 3**j** - 4**k**

Q8: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Direction ratios and direction cosines exercise (page 9)

Q9:

a) -3 : 2 : -1
b) 1 : 1 : 1
c) 2 : -1 : -1

Q10:

a) The modulus of the vector is 7. The direction cosines are $^2/_7$, $^3/_7$ and $^6/_7$
b) The modulus of the vector is $\sqrt{3}$. The direction cosines are $\frac{1}{\sqrt{3}}$, $\frac{-1}{\sqrt{3}}$ and $\frac{-1}{\sqrt{3}}$
c) The modulus of the vector is 6. The direction cosines are $^2/_3$, $^{-2}/_3$ and $^1/_3$

Basic skills exercise (page 12)

Q11: 3**i** - 3**k**

Q12: The vector addition gives $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

Q13: -5**i** + 2**j** - 5**k**

Q14: The vector is $\begin{pmatrix} 7 \\ 5 \\ -3 \end{pmatrix}$

Q15: -2 (2**i** - **j** + 3**k**) = -4**i** + 2**j** - 6**k**

These are scalar multiples of each other. The vectors are therefore parallel.

Q16: Let the vector be a**i** + b**j** + **k**

Then a**i** + b**j** + **k** = λ(2**i** - 4**j** + **k**)

Thus a = 2λ, b = -4λ and c = λ

But c = 3 and so λ = 3

The vector parallel to 2**i** - 4**j** + **k** with z = 3 is 3(2**i** - 4**j** + **k**) = 6**i** - 12**j** + 3**k**

Algebraic scalar product exercise (page 14)

Q17: $\mathbf{a} \bullet \mathbf{b}$ = -6 + 6 - 12 = -12

Q18: 2 - 9 + 2 = -5

Q19: The scalar product is -16

Answers from page 14.

Q20: $\mathbf{b} + \mathbf{c} = -4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$ = -8 - 2 = -10

$\mathbf{a} \bullet \mathbf{b}$ = -6 - 3 = -9 and $\mathbf{a} \bullet \mathbf{c}$ = -2 + 1 = -1

so $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ = -10 as required.

Q21: $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$ and

$\mathbf{b} \bullet \mathbf{a} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = b_1a_1 + b_2a_2 + b_3a_3$

But $a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3$ by the laws of algebra.

Thus $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

Geometric scalar product exercise (page 15)

Q22: 22.627

Q23: -10.825

Q24: $-6 = 4 \times 3 \times \cos\theta$

So $\cos\theta = -{}^{1}/_{2}$ which gives $\theta = 120°$

But the acute angle is always stated, so the angle is 60°

Q25: let $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{b} = 12\mathbf{i} + 5\mathbf{j}$

$\mathbf{a} \bullet \mathbf{b}$ = 16

a has length = 5

b has length = 13

So $\cos\theta = 16 / (5 \times 13) = {}^{16}/_{65}$

Thus $\theta = 75.75°$

Q26: $\overrightarrow{CA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$\overrightarrow{CB} = 3\mathbf{i} - 4\mathbf{k}$

cos ACB = $\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-5}{3 \times 5} = \frac{-1}{3}$ so angle ACB = 109.47°

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Perpendicular vectors exercise (page 16)

Q27: c = -3

Q28: c = 9

Q29: $\mathbf{a} \bullet \mathbf{b} = (-4 \times 2) + (-3 \times -2) + (2 \times 1) = 0$
Therefore the vectors are perpendicular.

Algebraic vector product exercise (page 17)

Q30: The answers are:

a) $-6\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
b) $-7\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}$
c) 0

Answers from page 17.

Q31: $\mathbf{b} + \mathbf{c} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$
$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = 9\mathbf{i} + 15\mathbf{j} + 3\mathbf{k}$
$\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
$\mathbf{a} \times \mathbf{c} = 6\mathbf{i} + 11\mathbf{j} + 4\mathbf{k}$
Thus $(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) = 9\mathbf{i} + 15\mathbf{j} + 3\mathbf{k}$ as required.

Vector product properties exercise (page 19)

Q32: $\mathbf{a} \times \mathbf{a} = (1 - 1)\mathbf{i} - (3 - 3)\mathbf{j} + (3 - 3)\mathbf{k} = 0$

Q33: $\mathbf{p} \times \mathbf{q} = (15 - 4)\mathbf{i} - (10 - 16)\mathbf{j} + (2 - 12)\mathbf{k} = 11\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$
$\mathbf{q} \times \mathbf{p} = (4 - 15)\mathbf{i} - (16 - 10)\mathbf{j} + (12 - 2)\mathbf{k} = -11\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$
Thus $\mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}$

Q34: $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

So $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

But $\mathbf{a} \times \mathbf{c} = \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix}$ and $\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 5 \\ -6 \\ -7 \end{pmatrix}$

So $(\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ as required.

Geometric vector product exercise (page 21)

Q35: Find the vector product. This will be perpendicular to both vectors.
It is -6**i** + 3**j** + 5**k**

Q36: **a x b** = 4 $\times$ 5 $\times$ sin 45° = 14.14 units

Q37: 12.728 square units

Q38: The area of the triangle is $\frac{1}{2}$ | **a x b** |
a x b = 2**i** + 4**j** - 5**k** and so the area is $^1/_2\sqrt{45}$ = 3.354 square units.

Q39: The area of the triangle is $3\sqrt{2}$ or 4.243 square units.

Scalar triple product exercise (page 23)

Q40: -4

Q41: 9

Q42: 20 cubic units

Q43: Let **a** = **i** + 2**j** + 3**k**
b = 4**i** - **j** + 2**k**
c = 2**i** - 5**j** - 4**k**
a • (**b x c**) = 0 and so the three points are coplanar; they lie in the same plane.

Answers from page 23.

Q44: **b x c** = -**i** + 2**j** + **k**
c x a = -**i** - **k**
a x b = **i** - 2**j** + **k**
a • (**b x c**) = -1 - 1 = -2
b • (**c x a**) = -2
c • (**a x b**) = 1 - 2 - 1 = -2
They are all equal.

Equation of a line exercise (page 28)

Q45: **r** = 3**i** + 4**j** + 5**k** + λ(-2**i** - **j** + 3**k**)

Q46: **r** = 3**i** - 2**j** + 4**k** + λ(4**i** - 3**j** - **k**)

Q47: The parametric equations are:

a) x = -1 - 2λ, y = 3 - λ and z = 4λ

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b) $x = 2 - \lambda$, $y = -1$ and $z = 4\lambda$

The symmetric equations are:

a) For line one $\frac{x+1}{-2} = \frac{y-3}{-1} = \frac{z-0}{4}$

b) For line two $\frac{x-2}{-1} = \frac{y+1}{0} = \frac{z-0}{4}$

Note the form of the middle term:

The zero is used in cases such as this since $y = -1$ and λ is not involved.

Q48:

a) For line one $\frac{x-0}{-1} = \frac{y+1}{1} = \frac{z-2}{-1}$

b) For line two $\frac{x-2}{-1} = \frac{y+2}{3} = \frac{z+2}{-1}$

Q49: The symmetric equations are $x = 2 + 3\lambda$, $y = 1 - \lambda$, $z = -3 - \lambda$

If $y = -1$ then $-1 = 1 - \lambda \Rightarrow \lambda = 2$ and the point is $(8, -1, -5)$

Q50: $a = (3, 4, 5)$ and $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ so this gives

$x = 3 + \lambda$, $y = 4 + \lambda$, $z = 5 + \lambda$

Q51: Solving each for λ gives $\frac{x-0}{-1} = \frac{y+2}{1} = \frac{z-3}{2}$

Equation from two points exercise (page 29)

Q52: Using the general equations

$\mathbf{r} = \mathbf{p} + \lambda(\mathbf{q} - \mathbf{p})$ and $\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}$

where $\mathbf{p} = (a_1, a_2, a_3) = (3, -1, 6)$ and $\mathbf{q} = (b_1, b_2, b_3) = (0, -3, -1)$ gives

parametric equations as $x = 3 - 3\lambda$, $y = -1 - 2\lambda$ and $z = 6 - 7\lambda$

symmetric equations as $\frac{x-3}{-3} = \frac{y+1}{-2} = \frac{z-6}{-7}$

Q53: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

This gives a direction vector of $\mathbf{i} + \mathbf{j} - \mathbf{k}$ by taking a multiple of it.

$\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Thus the parametric equations are

$x = 1 + \lambda$, $y = 2 + \lambda$, $z = -1 - \lambda$ and the symmetric equations are

$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z+1}{-1}$

Intersection points and angles exercise (page 34)

Q54: Let λ be the constant for line L_1 so

$\frac{x-2}{1} = \frac{y-2}{3} = \frac{z-3}{1} = \lambda$ so $x = 2 + \lambda$, $y = 2 + 3\lambda$ and $z = 3 + \lambda$

Let μ be the constant for line L_2 so

$\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-4}{2} = \mu$ so $x = 2 + \mu$, $y = 3 + 4\mu$ and $z = 4 + 2\mu$

If they intersect then

$2 + \lambda = 2 + \mu$

$2 + 3\lambda = 3 + 4\mu$

$3 + \lambda = 4 + 2\mu$

These solve to give $\lambda = \mu = -1$ and satisfy all equations.

The lines do meet at (1, -1, 2)

Q55: L_1 has parametric equations $x = -3 + \lambda$, $y = 2 + \lambda$, $z = 1 - 3\lambda$

L_2 has parametric equations $x = -4 - 2\mu$, $y = 1 + \mu$, $z = \mu$

Trying to solve the set of equations

$-3 + \lambda = -4 - 2\mu$

$2 + \lambda = 1 + \mu$

$1 - 3\lambda = \mu$ gives no consistent solution for λ and μ

Thus the equations do not intersect.

Q56: L_2 has parametric equations $x = -3 - \mu$, $y = 4 + \mu$, $z = -\mu$

Thus solving the equations

$-2 + 2\lambda = -3 - \mu$

$1 - 3\lambda = 4 + \mu$

$-1 + \lambda = -\mu$ gives $\lambda = -2$ and $\mu = 3$

Substituting in either set of equations gives the point of intersection as (-6, 7, -3)

The direction vectors are $\mathbf{d_1} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{d_2} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|} = \frac{-6}{\sqrt{14}\sqrt{3}}$

The angle of intersection is 157.79° but the acute angle is given and this is 22.21°

Q57: (1, 3, 2) and the angle is 72.95°

Q58: (9, 3, 0) and the angle is 77.47°

Q59: They intersect at the point (5, 9, 4) at an angle of 49.11°

Equation of a plane exercise (page 38)

Q60: $(\mathbf{r} - \mathbf{a}) \bullet \mathbf{n} = 0$ or $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$

Here $\mathbf{n} = 6\mathbf{i} - \mathbf{j} + {}^1/_3\mathbf{k}$

and $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

The equation is $6x - y + {}^1/_3 z = 6 - 2 + 1 = 5$

That is, $18x - 3y + z = 15$

Q61: $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two vectors in the plane. $\overrightarrow{AB}$ **x** $\overrightarrow{AC}$ will be perpendicular to both vectors and therefore to the plane. This vector product will give a normal to the plane.

$\overrightarrow{AB} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{AC} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

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so $\overrightarrow{AB}$ **x** $\overrightarrow{AC}$ = 7**i** - 5**j** - 4**k**

Thus a normal to the plane is **n** = (7, -5, -4)

A with position vector **a** = (1, 1, -1) is on the plane and the equation of the plane is **r** • **n** = **a** • **n** which gives 7x - 5y - 4z = 6

Q62: $\overrightarrow{AB}$ = -2**i** - 2**j** + 2**k**

$\overrightarrow{AC}$ = -**i** + 3**j** - 2**k**

$\overrightarrow{AB}$ **x** $\overrightarrow{AC}$ = -2**i** - 6**j** - 8**k**

The equation is x + 3y + 4z = 1 + 3 $\times$ 2 + 4 $\times$1 since the plane goes through A

That is, x + 3y + 4z = 11

Q63: Let y = λ and z = μ then x = - 3 + 2λ + 6μ

The parametric equation is **a** + λ**b** + μ**c** where **a** = -3**i**, **b** = 2**i** + **j** and **c** = 6**i** + **k**

Intersection and angles exercise (page 41)

Q64: Since it is parallel it takes the form x - 2y + z = d

A normal to the plane is **n** = **i** - 2**j** + **k** it passes through (1, -3, 2) so **a** = **i** - 3**j** + 2**k**

Since **r** • **n** = **a** • **n** then x - 2y + z = 1 + 6 + 2 = 9

The equation is x - 2y + z = 9

Q65: Let P_1 3x - 6y +9 z = -12 and P_2 = -x + 2y - 3z = 2

Let P_1 have normal **n** and P_2 have normal **m**.

Then -3**m** = **n** so P_1 and P_2 are either parallel or coincident.

The set of equations 3x - 6y + 9 z = -12 and -x + 2y - 3z = 2 have no solutions so P_1 and P_2 do not intersect.

Q66: A normal to the plane is **n** = 3**i** + **j** + **k**

a = **i** - **j** + 3**k**

Since **r** • **n** = **a** •**n** then 3x + y + z = 5

This is Cartesian form.

Q67: $\frac{x - 0}{1} = \frac{y + 2}{-1} = \frac{z - 1}{2}$ is the symmetric equations of the line of intersection of the planes.

Q68: The normals are not multiples of each other. Therefore the planes do intersect.

The equation of the line in parametric form is x = λ, y = $\frac{-13 + 5\lambda}{-3}$, z = $\frac{-14 + 7\lambda}{-3}$

Q69: The normals are **n** = **i** + **j** - 4**k** and **m** = 2**i** - 3**j** + 4**k**

n • **m** = 2 - 3 - 16 = -17

$|$ **n** $|$ = $\sqrt{18}$ and $|$ **m** $|$ = $\sqrt{29}$

cos θ = $\frac{-17}{\sqrt{18}\sqrt{29}}$ so θ = 138.1°

Taking the acute angle gives the angle between the two planes as 41.9°

Q70: The normals are **n** = 2**i** - **j** and **m** = **i** + **j** + **k**

n • **m** = 1, | n | = $\sqrt{5}$, | m | = $\sqrt{3}$

and so cos θ = $\frac{1}{\sqrt{15}}$ which gives θ = 75.04°

Intersection of three planes exercise (page 43)

Q71: Using Gaussian elimination gives the point (2, -2, 3)

Q72: Use Gaussian elimination.

They intersect at the point (2, 1, 3)

Q73: Using Gaussian elimination the system reduces to a zero line on the bottom. Thus the intersection is a line where z = -1 and x + y = 2. In symmetric form this equation of the line is $\frac{x+0}{1} = \frac{y-2}{-1} = \frac{z+1}{0}$

Q74: Using Gaussian elimination, the bottom row reduces to 0 = 42, which is impossible.

The three planes do not intersect.

Q75: The normals are **n** = **i** + **j** - 4**k** and **m** = 2**i** - 3**j** + 4**k**

n • **m** = 2 - 3 - 16 = -17

| **n** | = $\sqrt{18}$ and | **m** | = $\sqrt{29}$

cos θ = $\frac{-17}{\sqrt{18}\sqrt{29}}$ so θ = 138.1°

Taking the acute angle gives the angle between the two planes as 41.9°

Q76: The normals are **n** = 2**i** - **j** and **m** = **i** + **j** + **k**

n • **m** = 1, | n | = $\sqrt{5}$, | m | = $\sqrt{3}$

and so cos θ = $\frac{1}{\sqrt{15}}$ which gives θ = 75.04°

Lines, points and planes exercise (page 46)

Q77: The general point has coordinates (1 + 2λ, -1 - λ, 3λ)

Substituting in the plane gives 3 + 6λ - 2 - 2λ - 3λ = 5

and so λ = 4

The coordinates of the point of intersection are (9, -5, 12)

Q78: **d** = 2**i** + 3**j** + 6**k**

n = 10**i** + 2**j** - 11**k**

cos θ = $\frac{\mathbf{n} \bullet \mathbf{d}}{|\mathbf{n}||\mathbf{d}|}$ = $\frac{-40}{105}$and so θ = 112.4°

Thus the acute angle is 67.6°

The angle between the line and plane is 90 - 67.6 = 22.4°

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Q79: **n** = **i** + 2**j** - 2**k** and **m** = 5**i** - 2**j** - **k**

The vector product of these is along the direction of the line of intersection of the planes since it is perpendicular to both normals.

n x m = -6**i** - 9**j** - 12**k**

d = 2**i** + 3**j** + 4**k**

But **n x m** is a multiple of **d** and so it is parallel to it.

Q80: **n** = 2**i** + **j** - **k** and **m** = **i** + 2**j** + **k**

n x m = 3**i** - 3**j** + 3**k** thus a normal to the plane is **i** - **j** + **k**

2**i** + **j** - **k** is also on this plane

The equation of the plane is **r** • **n** = **a** • **n**

That is, x - y + z = 0

Review exercise in vectors (page 49)

Q81: **a x b** = -**i** - **j** - 3**k**

b x c = 6**i** - **j** + 4**k**

a • **b x c** = -3 - 4 = -7

Q82: The vector is -**i** + 2**j** + 6**k**

In parametric form the equations are

$x = 2 - \lambda$, $y = -3 + 2\lambda$ and $z = 1 + 6\lambda$

Q83: The normal is 2**i** + **j** + 3**k**

So the equation has the form 2x + y + 3z = c

The point (1, 2, 3) is on the plane thus 2 + 2 + 9 = c

The equation is 2x + y + 3z = 13

Q84: $x = \lambda$, $y = -\frac{1 + \lambda}{5}$ and $z = \frac{12 - 13\lambda}{5}$

Q85: Using Gaussian elimination gives the point as (3, -1, 2)

Advanced review exercise in vectors (page 50)

Q86: a) **u** • **v** = 9 and **u x v** = -6**i** -3**j** + 6**k**

b) i) The angle is 45 °

ii) Q is (1, -1, 2)

iii) L is the line with equations $\frac{x - 1}{-6} = \frac{y + 1}{-3} = \frac{z - 2}{6}$

It meets the xy-plane where z = 0. The point R = (3, 0, 0)

c) **a x b** = **c** so **a** is perpendicular to **c** and **b** is perpendicular to **c**

b x c = **a** so **a** is perpendicular to **b**

Since the vectors are mutually perpendicular

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a x **b** = **c** gives |**a**||**b**| = |**c**| and **b** x **c** = **a** gives |**b**||**c**| = |**a**|

so $\frac{|\mathbf{a}|}{|\mathbf{c}|} = |\mathbf{b}| = \frac{1}{|\mathbf{b}|}$

Therefore |**b**| = 1 That is, **b** is a unit vector

Q87: a) **p** • **q** = -9 and **p** x **q** = **i** + **j** + **k**

b) i) The point is (1, -1, 0)

b)ii) The equation of the line is $\frac{x - 0}{1} = \frac{y + \frac{1}{4}}{-\frac{3}{4}} = \frac{z - \frac{1}{4}}{-\frac{1}{4}}$

b)iii) It meets the yz-plane at (0, $^{-1}/_{4}$, $^{1}/_{4}$)

Q88: a) The equation is x - 3y - 2z = -5

b) The angle QPR is 11.74°

c) The angle is 138.11° i.e. the acute angle is 41.89

The angle between the line and the plane is 90 - 41.89° = 48.11°

Set review exercise (page 51)

Q89: The answers are only available on the web.

Q90: The answers are only available on the web.

Q91: The answer is available on the web.

2 Matrix algebra

Revision exercise (page 54)

Q1: The element a_{23} is 1

Q2: x = 2, y = -1 and z = 3

Q3:

1. (2 -3)
2. (-3, 2)

Q4: $(-a_2b_3 - a_3b_2)\mathbf{i} - (-a_1b_3 + a_3b_1)\mathbf{j} + (-a_1b_2 - b_1a_2)\mathbf{k}$

Adding matrices exercise (page 57)

Q5: $\begin{pmatrix} -2 & -5 \\ 4 & 2 \end{pmatrix}$

Q6: $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Q7: $\begin{pmatrix} -4 & 11 & -3 \\ 2 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$

Q8: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Subtracting matrices exercise (page 58)

Q9: $\begin{pmatrix} -3 & -3 \\ -5 & -2 \end{pmatrix}$

Q10: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Q11: $\begin{pmatrix} 2 & 4 & -3 \\ 2 & 1 & -4 \\ 1 & 0 & 4 \end{pmatrix}$

Q12: $\begin{pmatrix} -5 & 6 \\ -2 & -3 \end{pmatrix}$

Multiplication of a matrix by a scalar exercise (page 59)

Q13: $\begin{pmatrix} 4 & 2 & -6 \end{pmatrix}$

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Q14: λ = -1

Q15: $\lambda A = \begin{pmatrix} -4 & 0 & 4 \\ -8 & -4 & 8 \\ -12 & 8 & -16 \end{pmatrix}$

Multiplying matrices exercise (page 60)

Q16: The answer in this case is the 1 x 1 matrix (10)

Q17: The answer is the 3 x 4 matrix

$$\begin{pmatrix} -4 & -2 & -1 & -3 \\ 6 & 0 & 0 & -2 \\ 8 & -4 & 3 & -3 \end{pmatrix}$$

Q18: The following products can be formed: AB, BA, AC, AD, CB and CD.

The product matrices are

$AB = \begin{pmatrix} 0 & 2 \\ 7 & 7 \end{pmatrix}$, $BA = \begin{pmatrix} 8 & -3 & 1 \\ -1 & -4 & 3 \\ 9 & -6 & 3 \end{pmatrix}$, $AC = \begin{pmatrix} 7 & -3 & 7 \\ -4 & 4 & 9 \end{pmatrix}$

$AD = \begin{pmatrix} 5 & 5 & 1 & -7 \\ 9 & -1 & 4 & -2 \end{pmatrix}$, $CB = \begin{pmatrix} 1 & 11 \\ 4 & 0 \\ -7 & -10 \end{pmatrix}$, $CD = \begin{pmatrix} 9 & -5 & 4 & 4 \\ 3 & 3 & 1 & -5 \\ -11 & 3 & -5 & 0 \end{pmatrix}$

Q19: $\begin{pmatrix} -4 & -1 & -5 & 5 \\ 6 & 0 & 9 & -5 \\ 2 & 0 & 3 & 1 \\ -5 & 4 & 2 & 3 \end{pmatrix}$

Answers from page 61.

Q20:

1. These are not equal. There is no value for y which satisfies the equations.
2. Yes, these are equal with x = 1 and y = 6
3. These are not equal as the matrices are not of the same order.
4. Yes, these are equal with x = 3 and y = 0

2 x 2 determinant exercise (page 62)

Q21:

a) 5
b) 8
c) -1
d) 4
e) 0

3 x 3 determinant exercise (page 64)

Q22:

a) 2
b) 4
c) 1
d) 20

2 x 2 inverse exercise (page 65)

Q23:

a) $\frac{1}{5}\begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$

b) $\frac{1}{4}\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

c) $-1\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix}$

d) $\frac{1}{4}\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

e) $\frac{1}{12}\begin{pmatrix} 3 & -2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{6} \\ \frac{1}{4} & \frac{1}{6} \end{pmatrix}$

Answers from page 66.

Q24: The 2 x 2 identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 x 3 inverse exercise (page 67)

Q25:

1. $\begin{pmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}$

2. $\begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ -3 & -4 & 1 \end{pmatrix}$

3. $\begin{pmatrix} \frac{1}{9} & -\frac{4}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{5}{9} & \frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$

4. $\begin{pmatrix} \frac{6}{11} & \frac{8}{11} & -\frac{9}{11} \\ -\frac{2}{11} & \frac{1}{11} & \frac{3}{11} \\ -\frac{1}{11} & -\frac{5}{11} & \frac{7}{11} \end{pmatrix}$

5. $\begin{pmatrix} -\frac{5}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{11}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

Properties exercise (page 75)

Q26: $A + B = \begin{pmatrix} -2+(-2) & 3+1 \\ 0+2 & -2+1 \\ 1+1 & -1+2 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 2 & -1 \\ 2 & 1 \end{pmatrix}$

and $B + A = \begin{pmatrix} -2+(-2) & 1+3 \\ 2+0 & 1+(-2) \\ 1+1 & 2+(-1) \end{pmatrix} \begin{pmatrix} -4 & 4 \\ 2 & -1 \\ 2 & 1 \end{pmatrix}$ as required.

Q27: $AB = \begin{pmatrix} 6 & -5 \\ 2 & -2 \end{pmatrix}$ and $BA = \begin{pmatrix} -1 & -1 \\ 3 & 5 \end{pmatrix}$

Q28: $AB = \begin{pmatrix} 0 & 1 & 0 & -4 \\ -9 & 2 & 2 & 13 \end{pmatrix}$ and

$(AB)C = \begin{pmatrix} 0 & 1 & 0 & -4 \\ -9 & 2 & 2 & 13 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 20 \end{pmatrix}$

$BC = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ and $A(BC) = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 20 \end{pmatrix}$ as required.

Q29: $B + C = \begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix}$ and

$A(B + C) = \begin{pmatrix} 2 & -1 \\ -3 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}$

$AB = \begin{pmatrix} -5 & 0 \\ 6 & 3 \\ -1 & -3 \end{pmatrix}$ and $AC = \begin{pmatrix} 4 & 3 \\ -6 & -3 \\ 2 & 0 \end{pmatrix}$

so $AB + AC = \begin{pmatrix} -5 & 0 \\ 6 & 3 \\ -1 & -3 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ -6 & -3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}$ as required.

Q30: $A^T = \begin{pmatrix} -1 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$ and transposing this matrix gives $\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ 4 & 2 \end{pmatrix}$

So $(A^T)^T = A$

Q31: A + B = $\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 2 \\ -5 & 2 & 3 \end{pmatrix}$ and

$(A + B)^T = \begin{pmatrix} 1 & 0 & -5 \\ 3 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$

But $A^T = \begin{pmatrix} 2 & -2 & -3 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ and $B^T = \begin{pmatrix} -1 & 2 & -2 \\ 2 & 0 & 3 \\ -2 & 1 & 1 \end{pmatrix}$

so $A^T + B^T = \begin{pmatrix} 2 & -2 & -3 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 2 & -2 \\ 2 & 0 & 3 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 \\ 3 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$

as required.

Q32: $AB = \begin{pmatrix} 8 & -2 \\ 1 & -4 \end{pmatrix}$ so $AB^T = \begin{pmatrix} 8 & 1 \\ -2 & -4 \end{pmatrix}$

But $B^T = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $A^T = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix}$

which gives $B^TA^T = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ -2 & -4 \end{pmatrix}$ as required.

Q33: $AB = \begin{pmatrix} 2 & -3 \\ -10 & 3 \end{pmatrix}$ and det AB = -24

Thus $(AB)^{-1} = -\frac{1}{24}\begin{pmatrix} 3 & 3 \\ 10 & 2 \end{pmatrix}$

det A = 3 and det B = -8 so $A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$ and $B^{-1} = -\frac{1}{8}\begin{pmatrix} -1 & 3 \\ 2 & 2 \end{pmatrix}$

Thus $B^{-1}A^{-1} = -\frac{1}{8}\begin{pmatrix} -1 & 3 \\ 2 & 2 \end{pmatrix}$ times $\frac{1}{3}\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} = -\frac{1}{24}\begin{pmatrix} 3 & 3 \\ 10 & 2 \end{pmatrix}$ as required.

Answers from page 77.

Q34: The point under a rotation of -90° will go to the point (2, -5)

Let the matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

This gives two equations to solve:

$5a + 2b = 2$

$5c + 2d = -5$. The simplest solutions are a = 0, b = 1, c = -1 and d = 0

The matrix R such that RA = B is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix})$

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Answers from page 78.

Q35: $R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Rotation exercise (page 78)

Q36: $\begin{pmatrix} 6.428 \\ 4.866 \end{pmatrix}$

Q37: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.537 \\ 1.624 \end{pmatrix}$

This gives the two equations

$2\cos\theta° - \sin\theta° = 1.537$ and

$2\sin\theta° + \cos\theta° = 1.624$

Rearrange the first to give $\sin\theta° = 2\cos\theta° - 1.537$

Substitute this into the second to give $\cos\theta = 0.9396$

Thus $\theta = 20°$

Q38: RA = B so $A = R^{-1}RA = R^{-1}B$

As shown earlier $R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and the point is (-1, -2)

Reflection exercise (page 80)

Q39: $\begin{pmatrix} 0.6 \\ -3.1 \end{pmatrix}$

Q40: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

This gives the two equations

$-\cos 2\theta° - 2\sin\theta° = -2$ and

$-\sin 2\theta° + 2\cos\theta° = -1$

Rearrange the first to give $\cos 2\theta° = 2 - 2\sin 2\theta°$

Substitute this into the second to give $-\sin 2\theta + 4 - 4\sin 2\theta = -1$

Thus solving for θ gives $\theta = 45°$

Q41: The line y = -4x makes an angle of -76° with the x-axis.

If S is the rotation matrix and A is the original point with image B

Then SA = B so $A = S^{-1}SA = S^{-1}B$

$S^{-1} = -1\begin{pmatrix} -\cos(-152) & -\sin(-152) \\ -\sin(-152) & \cos(-152) \end{pmatrix} = \begin{pmatrix} \cos(-152) & \sin(-152) \\ \sin(-152) & -\cos(-152) \end{pmatrix}$

and the point is (0.9, 0.5)

Scaling exercise (page 82)

Q42: The matrix is $\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$ and the image of the point is (-9, 1)

Q43: Let the matrix be $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

Then $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

This gives two simple equations which solve to give a = 2 and b = 0.5

The matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$

Q44: Let the matrix be $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

Then 3a = -1 and 4b = 0

Thus a = $-^1/_3$ and b = 0

The matrix is $\begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix}$

General transformation exercise (page 83)

Q45: Answer:

Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ gives the equations

1) -3a + 2b = 6 and

2) -3c + 2d = 7

Also $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ -13 \end{pmatrix}$ gives the equations

3) 5a + 4b = -8 and

4) 5c + 4d = -13

Solving the set of equations, 1) and 3) gives a = $^{-20}/_{11}$ and b = $^3/_{11}$

Similarly the set of equations 2) and 4) gives c = $^{-27}/_{11}$ and d =$^{-2}/_{11}$

The matrix is $\begin{pmatrix} -\frac{20}{11} & \frac{3}{11} \\ -\frac{27}{11} & -\frac{2}{11} \end{pmatrix}$

Q46: Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

This gives equations 2a - 3b = -8 and -a - 4b = -7

These solve to give a = -1 and b = 2

Similarly the equations 2c - 3d = 16 and - c - 4d = 3 solve to give c = 5 and d = -2

The matrix is $\begin{pmatrix} -1 & 2 \\ 5 & -2 \end{pmatrix}$

Q47: The point is (18, -16)

Q48: The matrix is $\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$

Q49: Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Take the two points on the first line and find two sets of equations to solve. The two sets here are

1) $2a + 2b = 0$ and $3a + 4b = -2$

2) $2c + 2d = 8$ and $3c + 4d = 13$

The matrix is $\begin{pmatrix} 2 & -2 \\ 3 & 1 \end{pmatrix}$

Review exercise (page 88)

Q50:

a) $\begin{pmatrix} 4 & 17 \\ -14 & 7 \end{pmatrix}$

b) $\begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 11 & -9 \\ 15 & 9 \end{pmatrix}$

d) 36

Q51:

a) $\begin{pmatrix} -8 & 2 \\ -14 & -4 \end{pmatrix}$

b) $\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}$

c) $\begin{pmatrix} 1 & -3 \\ -9 & -7 \end{pmatrix}$

d) -31

Advanced review exercise (page 88)

Q52: $2A^2 = A + I$ rearranges to give $A(2A - 1) = I$

Thus $2A - 1$ is the inverse matrix and so A is invertible. $A^{-1} = 2A - 1$

Q53: $A^2 - 4I = 0$

So $(A - 2I)(A + 2I) = 0$ thus $A = 2I$ or $A = -2I$ i.e. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

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Q54: The rotation matrix to use is $\begin{pmatrix} \cos(-60) & -\sin(-60) \\ \sin(-60) & \cos(-60) \end{pmatrix}$ followed by the scaling matrix $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$. The reflection is straightforward from there and the image of the point has coordinates (3.196,9.696) to one decimal place.

Q55: Since A is orthogonal $A^T = A^{-1}$ and $AA^T = I$

Thus $(AA^T)^{-1} + A(3A^T - A^{-1}) = I^{-1} + A(2A^{-1}) = I + 2I = 3I$

Set review exercise (page 89)

Q56: The answer is only available on the web.

Q57: The answer is only available on the web.

3 Further sequences and series

Revision Exercise (page 92)

Q1: 32

Q2: $1 + \frac{(-1)}{1!}x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \dots$

Q3: $f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

Q4:

u_0	5
u_1	12.5
u_2	17.75
u_3	21.425
u_4	23.9975

As $n \Rightarrow \infty$, $u_n \Rightarrow 30$, $L = 30$

Answers from page 93.

Q5: $S_{10} = 2.718282 \approx e$

Exercise 1 (page 95)

Q6:

a) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$

b) $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \dots$

c) $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \dots$

Answers from page 96.

Q7: $f^{(4)}(0) = (4 \times 3 \times 2 \times 1)a_4 = 4!a_4$

Answers from page 99.

Q8: We obtain the following results

$f(x) = \ln x$	$f(0) = \ln 0$
$f^{(1)}(x) = \frac{1}{x}$	$f^{(1)}(0) = \frac{1}{0}$
$f^{(2)}(x) = \frac{-2}{x^2}$	$f^{(2)}(0) = \frac{-2}{0}$

$f(x) = \ln x$ and its derivatives are undefined at $x = 0$ therefore we cannot find a Maclaurin series expansion for $\ln x$

Q9: We obtain the following results

$f(x) = \sqrt{x} = x^{1/2}$ $\qquad f(0) = \sqrt{0}$

$f^{(1)}(x) = \frac{1}{2x^{1/2}}$ $\qquad f^{(1)}(0) = \frac{1}{0}$

$f^{(2)}(x) = -\frac{1}{4x^{3/2}}$ $\qquad f^{(2)}(0) = -\frac{1}{0}$

The derivatives of f (x) = $\sqrt{x}$ are undefined at x = 0 therefore we cannot find a Maclaurin series expansion for $\sqrt{x}$

Q10: We obtain the following results

$f(x) = \cot x = \frac{1}{\tan x}$ $\qquad f(0) = \frac{1}{0}$

$f^{(1)}(x) = -\operatorname{cosec}^2 x = -\frac{1}{\sin^2 x}$ $\qquad f^{(1)}(0) = -\frac{1}{0}$

$f^{(2)}(x) = 2\cot x \operatorname{cosec}^2 x = 2\frac{\cos x}{\sin^3 x}$ $\qquad f^{(2)}(0) = \frac{2}{0}$

f (x) = cot x and its derivatives are undefined at x = 0 therefore we cannot find a Maclaurin series expansion for cot x

Exercise 2 (page 100)

Q11: **a)** $e^{5x} = 1 + \frac{5x}{1!} + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \frac{(5x)^4}{4!} + \frac{(5x)^5}{5!} + \ldots$

b) $\sin 2x = \frac{2x}{1!} - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{8!} - \ldots$

c) $\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \ldots$

d) $\cos(-x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$

e)

Q12:

a)

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \ldots$$

b) $e^{ix} = \cos x + i \sin x$

Exercise 3 (page 102)

Q13:

a) $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4$

b) $1 - 5x + 15x^2 - 35x^3 + 70x^4$

c) $1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4$

Q14:

a) $16 + 32x + 24x^2 + 8x^3 + x^4$

b) $1 - 6x + 12x^2 - 8x^3$

Q15:

a) $\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3$

b) $-\frac{1}{8} - \frac{3}{16}x - \frac{3}{16}x^2 - \frac{5}{32}x^3$

Exercise 4 (page 104)

Q16: a), b) and **i)** all diverge.

c) converges to 20

d) converges to 8

e) converges to 5

f) converges to 4

g) converges to $3^1/_3$

h) converges to 2.5

Q17: $-1 < a < 1$

Q18:

a) converges to 12.5

b) converges to 0.75

c) converges to -0.25

d) converges to -5

Q19: Changing b will not change whether a recurrence relation converges or not. It will change the value of the limit and may also alter the rate at which the recurrence relation converges to this limit.

Q20: In each case the recurrence relation converges to 7.5

Q21: It does not seem to matter what value we choose for u_0. The recurrence relation always converges to the same limit.

Answers from page 105.

Q22:

a) $-^4/_3$ **b)** 20 **c)** $3^1/_3$

Answers from page 106.

Q23: **a)** $u_{n+1} = -0.3u_n + 7$

b) $u_{n+1} = 2u_n + 5$

c) $u_{n+1} = 0.4u_n + 6$

d) $u_{n+1} = 0.8u_n + 4$

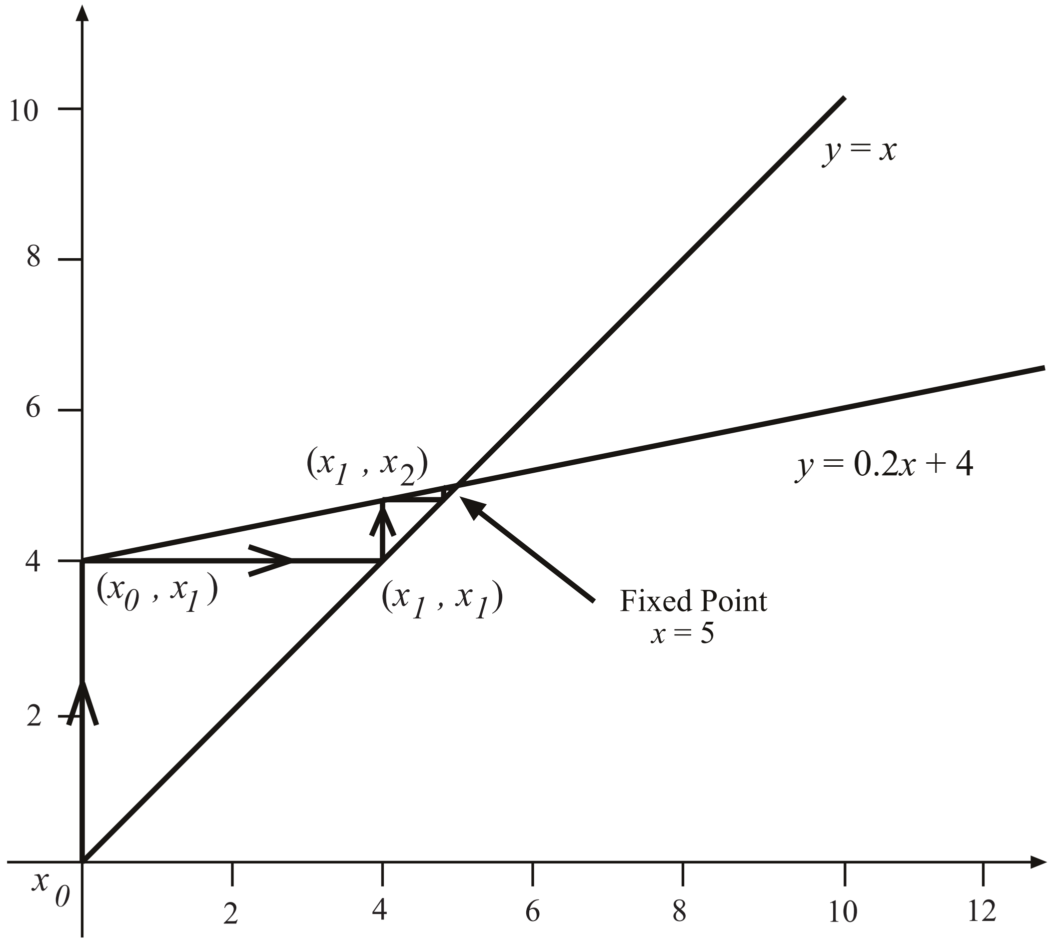
e) $u_{n+1} = -0.2u_n + 3.6$

f) $u_{n+1} = 1.5u_n + 15$

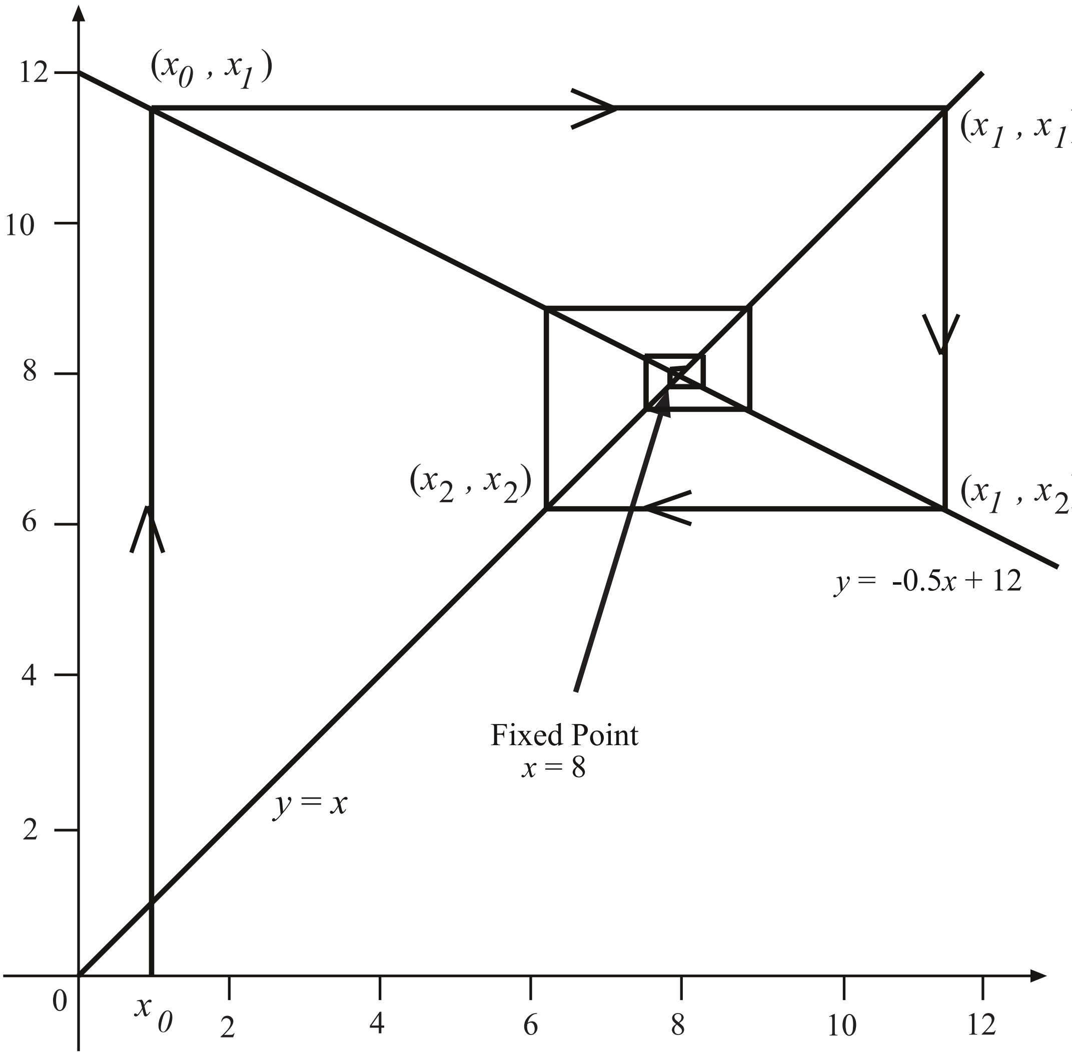
Q24: **a), c), d)** and **e)**

Staircase and Cobweb diagrams (page 109)

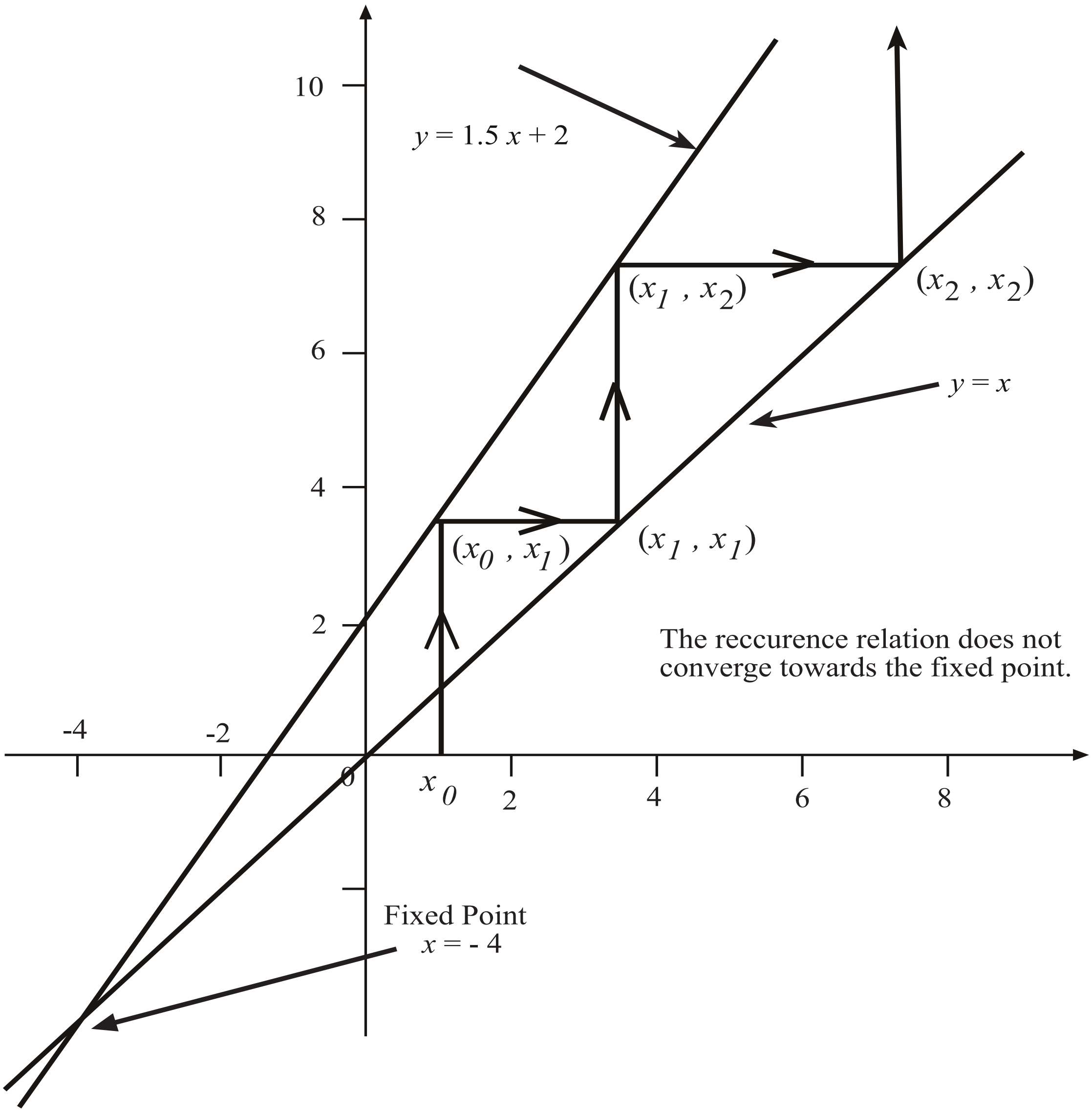
Q25: **a)**



b)

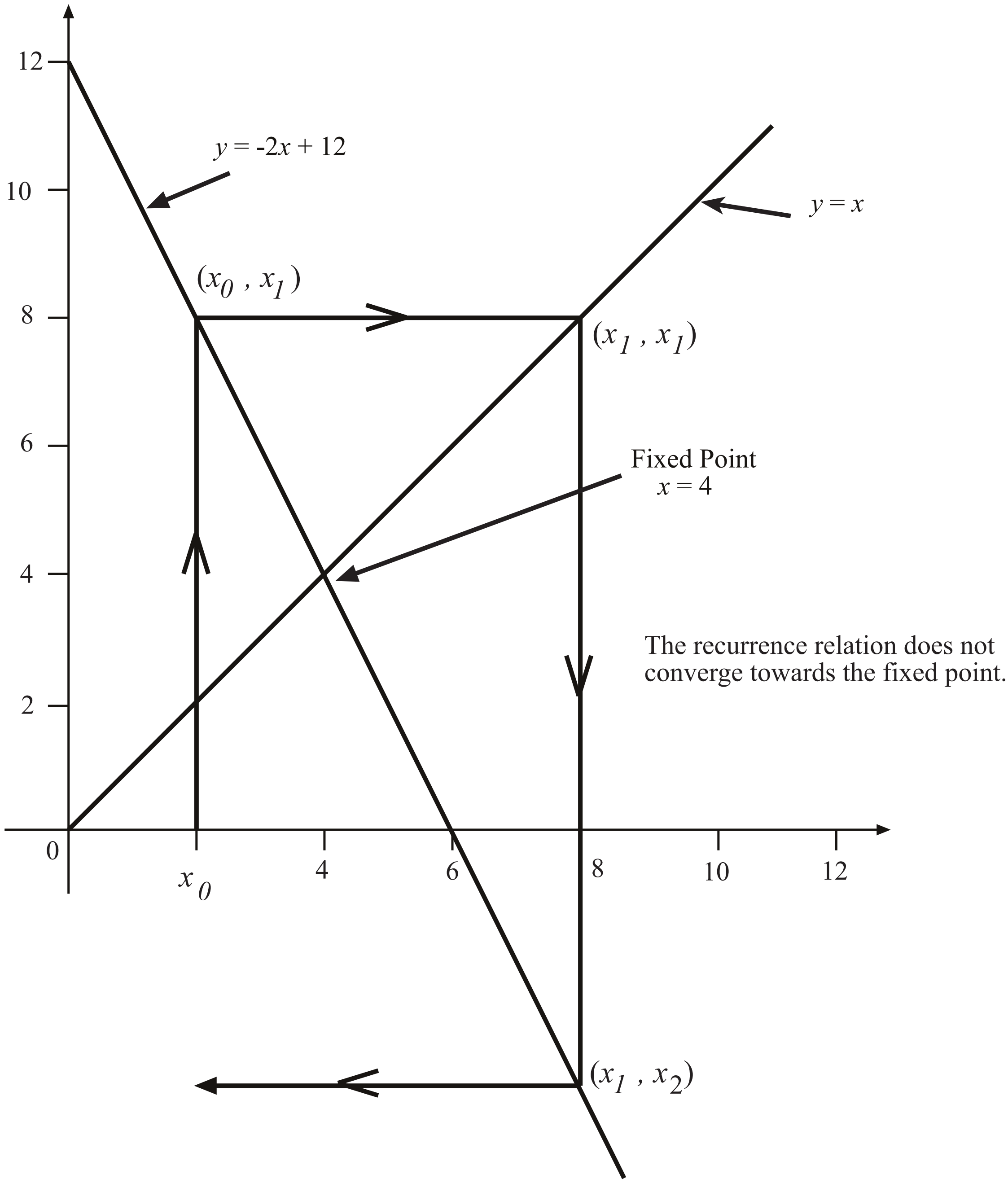


c)



10
8
6
4
2
-4
-2
0
x_0
2
4
6
8
$y = 1.5x + 2$
(x_1, x_2)
(x_2, x_2)
$y = x$
(x_0, x_1)
(x_1, x_1)
The reccurence relation does not converge towards the fixed point.
Fixed Point
$x = -4$

d)



Testing Convergence (page 109)

Q26: This time the sequence converges to 0.523976 This is the root that we found already. We have not found a more accurate value for the root near 2.1

Q27:

	$x_0 = -2.7$	$x_0 = 0.5$	$x_0 = 2.1$
$x_{n+1} = \frac{x_n^3 + 3}{6}$	Diverges	Converges to 0.523976	Converges to 0.523976
$x_{n+1} = \frac{3}{6 - x_n^2}$	Converges to 0.523976	Converges to 0.523976	Converges to 0.523976
$x_{n+1} = x_n^3 - 5x_n + 3$	Diverges	Diverges	Diverges
$x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 6}$	Converges to -2.669079	Converges to 0.523976	Converges to 2.145103

Q28: Of the iterative schemes given, only $x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 6}$ gave values for all three roots.

Exercise 5 (page 112)

Q29: $x_{n+1} = e^{-x_n}$ gives the solution 0.567143 (to 6 decimal places)

Q30: $x_{n+1} = 2 - \sin x_n$ gives the solution 1.106060 (to 6 decimal places)

Q31:

$x = {}^1/_2 - x^3$ gives the solution 0.423854

$x = \left(\frac{1}{2} - x\right)^{\frac{1}{3}}$ does not give the solution. It oscillates between the values -0.845263 and 1.103915

Q32:

$x = e^x - 3$ has only one solution, -2.947531 (to 6 decimal places)

$x = \ln(x + 3)$ has only one solution, 1.505241 (to 6 decimal places)

Exercise 6 (page 114)

Q33: **a)** $x_0 = 1.7$

b) 1.749031

Q34: **a)** $x_0 = -1.8$

b) The iterative scheme $x_{n+1} = (x - 4)^{1/3}$ will give the solution -1796322 (to 6 decimal places)

Q35:

a) $x_0 = 2.2$

b) The iterative scheme $x_{n+1} = 3 - \ln x_n$ will give the solution 2.207040 (to 6 decimal places)

Q36:

a) $x_0 = 0.2$ or $x_0 = 1.6$

b) $x_{n+1} = (5x - 1)^{1/4}$ will give the solution 1.637306 (to 6 decimal places)1.637306 (to 6 decimal places) and $x_{n+1} = {}^1/_5(x^4 + 1)$ will give the solution 0.200322 (to 6 decimal places)

Exercise 7 (page 117)

Q37:

a)

$g(x) = (4x + 5)^{1/3}$

$g'(x) = \frac{4}{3}(4x + 5)^{-2/3}$

$g'(2) = 0.24$ approx

$|g'(2)| < 1$ therefore the iterative process will converge to the solution.

b)

$g(x) = \frac{1}{4}\left(x^3 - 5\right)$

$g'(x) = \frac{3}{4}x^2$

$g'(2) = 3$

$|g'(2)| > 1$ therefore the iterative process will diverge.

Q38:

a) $g'(-1.9) = 24.9$, $g'(-0.3) = 0.039$ and $g'(2.1) = -24.9$ Therefore we should only expect the given iterative process to converge to the solution near -0.3

b) -0.254102 (to 6 decimal places)

Q39:

a)

For (A) $g'(-1.8) = -3.88$ approx

For (B) $g'(-1.8) = 0.27$ approx

For (C) $g'(-1.8) = 0.07$ approx

(A) will not converge to the required root but (B) and (C) will.

b) -1.752332 (to 6 decimal places).

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Exercise 8 (page 119)

Q40:

a)

$$g(x) = \frac{2}{x+4} = 2(x+4)^{-1}$$

$$g'(x) = -2(x+4)^{-2}$$

$$= \frac{-2}{(x+4)^2}$$

For α close to zero $g'(\alpha) = \frac{-2}{(\alpha+4)^2} \neq 0$

Therefore this iterative process is first order.

b)

$g(x) = 1 - \sin x$

$g'(x) = -\cos x$

We are given that $0 < \alpha < 1$ radians therefore $g'(\alpha) \neq 0$

($\cos x = 0$ when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ radians and $\frac{\pi}{2}, \frac{3\pi}{2} > 1$)

Q41:

$$g(x) = \frac{1}{2}x\left(3 - 5x^2\right)$$

$$= \frac{3}{2}x - \frac{5}{2}x^3$$

$$g'(x) = \frac{3}{2} - \frac{15}{2}x^2$$

At $\alpha = \frac{1}{\sqrt{5}}$, $g'\left(\frac{1}{\sqrt{5}}\right) = \frac{3}{2} - \frac{15}{2} \times \frac{1}{5} = 0$

Therefore this iterative process has second order convergence.

Review exercise in further sequences and series (page 122)

Q42: $1 + 3x + \frac{9x^2}{2}$

Q43: -2.104

Advanced review exercise in further sequences and series (page 122)

Q44: $1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{5}{8}x^4$

Q45: $\frac{1}{3^2} + \frac{2}{3^3}x + \frac{1}{3^3}x^2 + \frac{4}{3^5}x^3$

Q46:

a)

$g'(x) = -2e^{-x}$

$|g'(1)| \approx |-0.736| < 1$

Therefore the iterative process will converge to the solution near x = 1

α = 0.853 (to 3 decimal places).

b) $g'(\alpha) = \frac{-2}{e^{-\alpha}} \neq 0$

Therfore the convergence is first order.

Q47:

a)

x_0	2
x_1	1.83333
x_2	1.81726
x_3	1.81712
x_4	1.81712

To three decimal places α = 1.817

b)

$g'(x) = \frac{2}{3} - \frac{4}{x^3}$

$g'(6^{1/3}) = \frac{2}{3} - \frac{4}{6} = 0$

Therefore the convergence is second order.

Set review exercise in further sequences and series (page 123)

Q48: This answer is only available on the course web site.

Q49: This answer is only available on the course web site.

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4 Further Ordinary Differential Equations

Revision exercise (page 126)

Q1: $f'(x) = 2x \sin x + x^2 \cos x$

Q2: ${}^{dy}/_{dx} = 4e^{4x}$

Q3: $3 \ln(x) + C$

Q4: x^3

Q5: $\pm 7i$

Q6: $-2 \pm 3i$

Exercise 1 (page 130)

Q7: $y = e^{3x} + C e^{2x}$

Q8: $y = x^3 + {}^{C}/_{x}$

Q9: $y = \frac{1}{2}e^{x/2} + Ce^{-x/2}$

Q10: $y = \frac{\sin x}{x^2} + \frac{C}{x^2}$

Q11: $y = -\frac{2}{x} + Cx$

Q12: $y = -\cos x + \frac{\sin x}{x} + \frac{C}{x}$

Q13:

$$y = 2x - \frac{1}{x^2}$$

Q14: $y = \frac{2}{5} + \frac{1}{5}e^{(1 - 5x)}$

Q15: $y = e^{\sin(x)}\left(x^2 - \pi^2\right)$

Q16: $y = e^{1/x}(x - 1)$

Q17: $y = 2 - e^{-x^3}$

Exercise 2 (page 135)

Q18: 3.74 kg

Q19: 4 hours 25 minutes

Q20: 13.5 g/litre

Q21:

a) 731.7 kg

b) Approximately 103 minutes.

c) As $t \Rightarrow \infty$ then $M = 1500 - 1400e^{-0.01t} \Rightarrow 1500$ since $e^{-0.01t} \Rightarrow 0$

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Q22: $v\,(t) = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$

Q23: **a)** $c\,(t) = \frac{G}{100Vk}\left(1 - e^{-kt}\right) + 100e^{-kt}$

b) As $t \Rightarrow \infty$, $e^{-kt} \Rightarrow 0$. Therefore the limit of concentration in the blood is $\frac{G}{100Vk}$

Save the fish (page 136)

The formula for the correct answer is

$w = -\frac{5000 \times \ln 0.6}{t}$

Exercise 3 (page 142)

Q24: $y = Ae^{-2x} + B\,e^{3x}$

Q25: $y = A\,e^{-x} + B\,e^{-5x}$

Q26: $y = (A + Bx)\,e^{-3x}$

Q27: $y = e^{-2x}\,(A \cos 3x + B \sin 3x)$

Q28: $y = (A + Bx)e^{5x}$

Q29: $y = e^{-2x}\,(A \cos 2x + B \sin 2x)$

Q30: $y = A\,e^{-3x} + Be^{x/2}$

Q31: $y = (A + Bx)e^{x/2}$

Q32: $y = (A + Bx)e^{-2x/3}$

Q33: $y = e^{-x/2}\left(A \cos(\frac{3}{2}x) + B \sin(\frac{3}{2}x)\right)$

Q34: $y = e^{-2x}\left(A \cos\sqrt{2}x + B \sin\sqrt{2}x\right)$

Q35: $y = Ae^{-3.41x} + Be^{-0.59x}$

Exercise 4 (page 145)

Q36: $y = -2e^{-3x} + 2e^{2x}$

Q37: $y = 3xe^{2x}$

Q38: $y = e^{-x}(2 \cos x + 5 \sin x)$

Q39: $y = 5e^{x-1} + 2e^{5(x-1)}$

Q40: $y = (5 + 2x)e^{4x - 4}$

Q41: $y = e^{3(x - \pi)}\,(\cos 2x + 4 \sin 2x)$

Q42: $x(t) = 40 \cos 4t - 3 \sin 4t$

When $t = 5$, $x = 13.6$ cm

Q43: $x = 0.25e^{-t} - 0.75te^{-t}$

Q44: $x = e^{-0.05t}(0.7 \cos 0.35t + 0.1 \sin 0.35t)$

Exercise 5 (page 148)

Q45: **a)** When $y = e^{4x}$ then

$\frac{dy}{dx} = 4e^{4x}$ and $\frac{d^2y}{dx^2} = 16e^{4x}$

Therefore $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 16e^{4x} - 24e^{4x} + 9e^{4x} = e^{4x}$

Hence e^{4x} is a particular integral for the equation and we can write $y_p = e^{4x}$

b) $y = e^{4x} + A e^{3x} + Bx e^{3x}$

Q46: **a)** When $y = e^{3x}$ then

$\frac{dy}{dx} = 3e^{3x}$ and $\frac{d^2y}{dx^2} = 9e^{3x}$

Therefore $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 9e^{4x} + 6e^{4x} + 5e^{4x} = 20e^{4x}$

Hence e^{3x} is a particular integral for the equation and we can write $y_p = e^{3x}$

b) $y = e^{3x} + e^{-x}(A \cos 2x + B \sin 2x)$

Exercise 6 (page 151)

Q47:

a) $y_c = A e^{3x} + Bx e^{3x}$

b) $y_p = e^{4x}$

c) $y = e^{4x} + A e^{3x} + bx e^{3x}$

Q48:

a) $y_c = A e^{x} + B e^{4x}$

b) $y_p = 2x^2 + 5x + 6$

c) $y = 2x^2 + 5x + 6 + A e^{x} + B e^{4x}$

Q49:

a) $y_c = A e^{-2x} + Bx e^{-2x}$

b) $y_p = 3 \sin x - 4 \cos x$

c) $y = 3 \sin x - 4 \cos x + A e^{2x} + Bx e^{2x}$

Q50:

a) $y_c = e^{-x}(A \cos 3x + B \sin 3x)$

b) $y_p = 1 - x$

c) $y = 1 - x + e^{-x}(A \cos 3x + B \sin 3x)$

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Q51:

a) $y_c + A\ e^{3x} + Bx\ e^{3x}$

b) $y_p = 2$

c) $y = 2 + A\ e^{3x} + Bx\ e^{3x}$

Q52:

a) $y_c = e^{-x}\ (A \cos 4x + B \sin 4x)$

b) $y_p\ {}^{1}/_{2}\ e^{x}$

c) $y = {}^{1}/_{2}\ e^{x} + e^{-x}(A \cos 4x + B \sin 4x)$

Q53:

a) $y_c = Ae^{-5x} + Be^{2x}$

b) $y_p = 3x\ e^{2x}$

c) $y = 3x\ e^{2x} + A\ e^{-5x} + B\ e^{2x}$

Q54:

a) $y = A\ e^{x} + B\ e^{4x}$

b) $y_p = \cos 3x + 3 \sin 3x$

c) $y = \cos 3x + 3 \sin 3x + A\ e^{x} + B\ e^{4x}$

Q55:

a) $y_c = A\ e^{-2x} + B\ e^{x}$

b) $y_p = \sin x - 6 \cos x$

c) $y = \sin x - 6 \cos x + A\ e^{-2x} + B\ e^{x}$

Q56:

a) $y = 3 - 6x + A\ e^{-x} + B\ e^{2x}$

b) $y = 2\ e^{3x} + A\ e^{-x} + B\ e^{2x}$

c) $y = 3 - 6x + 2e^{3x} + A\ e^{-x} + B\ e^{2x}$

Review exercise in further ordinary differential equations (page 160)

Q57: $y = {}^{1}/_{4}e^{-x} + Ce^{-5x}$

Q58: $y = e^{-x^2}\ (\sin x + C)$

Advanced review exercise in further ordinary differential equations (page 160)

Q59: $y = x^4 + \frac{4}{x^2}$

Q60:

a) $I(x) = e^{\int \tan x\,dx}$ so we need to find $\int \tan x\,dx$ first.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$

Let u cosx then $\frac{du}{dx} = -\sin x$

We now have
$$\int \frac{\sin x}{\cos x}dx = \int \frac{\sin x}{u}\frac{dx}{du}du$$
$$= \int -\frac{1}{u}du$$
$$= -\ln u$$
$$= -\ln(\cos x)$$
$$= \ln(\cos x)^{-1}$$

Hence $I(x) = e^{\int \tan x\,dx} = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1} = \frac{1}{\cos x}$

b) $y = \sin x \cos x + \sqrt{2} \cos x$

Q61: $x = e^{-t}(A \cos 2t + B \sin 2t)$

Q62: $y = (3 + 5x)e^{5x}$

Q63:

a) $y = 4 \cos x - 8 \sin x + A e^{x} + Be^{3x}$

b) $y = 8 \cos x + 4 \sin x + A e^{x} + Be^{3x}$

c) $y = 12 \cos x - 4 \sin x + A e^{x} + Be^{3x}$

Set review exercise in further ordinary differential equations (page 161)

Q64: This answer is only available on the course web site.

Q65: This answer is only available on the course web site.

5 Further number theory and further methods of proof

Answers from page 164.

Q1: In the first topic on number theory the following diagram and definitions were given to explain the connection.

definition

$\downarrow$

conjecture

$\downarrow$

proof

$\downarrow$

theorem

A conjecture is a precise, unambiguous statement for which a convincing argument is needed.

A proof is a logically rigorous and complete argument that a mathematical statement is true.

A theorem is a mathematical conjecture which has been proved.

Q2: 16.25

S_4 is the notation for the sum of the first four terms.

This is the answer to (1 + 3) + (2 + 1.5) + (3 + 1) + (4 + 0.75)

The sigma sign has a starting value (the lower limit) and an ending value (the higher limit) to indicate which values of r should be used.

Q3: Using simple algebraic manipulation

$$4x^3 + 12xy + \frac{5x^2}{y} + 15 = 4x^3 + \frac{5x^2}{y}$$

$$\Rightarrow\ 12xy + 15 = 0$$

$$\Rightarrow xy = -\frac{5}{4}$$

Q4: The terms mean:

1. Implies (from left to right).
2. Implied by (from right to left).
3. If and only if (implication in both directions).

Symbol exercise (page 166)

Q5:

1. The correct symbol is $\Leftarrow$. There are many clear, colourless liquids which are certainly not drinking water.

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2. The correct symbol is $\Rightarrow$. x = 3 - 4i is a complex number which does not imply that $x^2 = -2$
3. The correct symbol is $\Leftarrow$. R may be a square singular matrix.
4. The correct symbol is $\Rightarrow$. 15 is an odd number which is not prime.

Q6:

1. If P is 'x + 8 = 4' and Q is 'x is a negative number' then P $\Rightarrow$ Q
2. If P is ' x $\geq$ 0' and Q is 'x + 1 is positive' then P $\Rightarrow$ Q
3. Careful: if Q is sin θ = 0 and P is $\theta = \pi$ then P $\Rightarrow$ Q

Equivalence exercise (page 168)

Q7: The answers are:

1. Equivalence is correct here with the assumption that step parents are not being considered.

P = Joe is Amy's father	Q = Amy is Joe's daughter	P $\Leftrightarrow$ Q
1	1	1
1	0	0
0	0	1
0	1	0

2. Equivalence is incorrect here. The correct sign is $\Rightarrow$ If x is an even number, this does not imply that it is divisible by 6: for example, 4 is even.
3. Equivalence is incorrect here. None of the implication signs necessarily apply here. I study Maths does not imply that I am good at Maths. Similarly I am good at Maths does not imply that I study Maths.
4. Equivalence is incorrect. In this last case the correct sign is $\Leftarrow$

   Take x = 9; x here is odd and does not imply that x is a prime since 9 is not a prime. However any prime greater than 2 is odd and so the implication works from Q to P where Q is the proposition that x is a prime >2 and P is the proposition that x is odd.

Answers from page 168.

Q8: The truth table is simply

P	$\neg$P
1	0
0	1

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Negation exercise (page 169)

Q9:

1. $\neg$P:$\sqrt{x}$ is irrational.
2. $\neg$P: I dislike Maths.
3. $\neg$P: x is not a negative real. Note that x could be zero and this rules out putting the word positive into the answer.
4. $\neg$P: $x \leq 0$, $x \in \mathbb{Z}$.

Converse exercise (page 170)

Q10: 'If I am at home then it is Thursday.' The proposition P is 'it is Thursday' and the proposition Q is 'I will be at home'. So S : P $\Rightarrow$ Q and the converse is Q $\Rightarrow$ P

Q11:

1. P: I am over fifty. Q: I am over forty.
   So Q $\Rightarrow$ P states 'I am over forty $\Rightarrow$ I am over fifty'!
2. P: It rains. Q: It is cloudy.
   So Q $\Rightarrow$ P, meaning if Q then P reads 'If it is cloudy then it rains'.
3. P: I work. Q: I am at Heriot Watt University.
   So Q $\Rightarrow$ P meaning Q only if P reads 'I am at Heriot Watt University only if I am working'.
4. P: It is dry. Q: Joyce cycles to school.
   So Q $\Rightarrow$ P, meaning P if Q reads 'It is dry if Joyce cycles to school'. Remember in this case that the phrase Q if P means P $\Rightarrow$ Q and this is why P and Q are defined as shown.

Contradiction exercise (page 173)

Q12: Assume that the conjecture is false and so $\sqrt{2} + \sqrt{3} \geq \sqrt{10}$

Square both sides to give

$2 + 2\sqrt{6} + 3 \geq 10 \Rightarrow 2\sqrt{6} \geq 5$

Square both sides to give

$24 \geq 25$ which is a contradiction.

Thus the assumption that $\sqrt{2} + \sqrt{3} \geq \sqrt{10}$ is untrue and the conjecture is true. That is, $\sqrt{2} + \sqrt{3} < \sqrt{10}$

Q13: Assume the conjecture is false and make a new assumption that $m^2 = 14$ but m is rational.

Therefore $m = \frac{p}{q}$ where p and q are relatively prime.

So $14q^2 = p^2 \Rightarrow p^2$ is even since $14q^2 = 2 \times (7q^2)$

By the recently-proved conjecture (see the example) that 'if m^2 is even then m is even'

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then **p is even** and p = 2k for some integer k

Thus $14q^2 = 4k^2 \Rightarrow 7q^2 = 2k^2$

Using the same argument again

$7q^2 = 2m^2 \Rightarrow q^2$ is even and so **q is even**.

Thus both p and q are even but both were assumed to be relatively prime. This is a contradiction and the assumption that m is rational is false.

The original conjecture that if $m^2 = 14$ then m is not a rational number is therefore true.

Q14: Assume that the conjecture is untrue and make the assumption that x + y is irrational but both x and y are rational.

Thus $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers a, b, c and d

$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

This is rational and contradicts the assumption that if x + y is irrational then both x and y are rational.

The conjecture that if x + y is irrational then at least one of x, y is irrational is true.

Q15: Assume that the conjecture is untrue and make a new assumption that n^2 is a multiple of 5 but n is not a multiple of 5. There are therefore four possibilities to consider.

n takes one of these forms:

a) 5k + 1
b) 5k + 2
c) 5k + 3
d) 5k + 4

Taking each in turn:

a) Suppose n = 5k + 1 then
   $n^2 = (5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1$
   Which is of the form 5t + 1 for another integer t
   Thus n^2 is not a multiple of 5 which is false and the new assumption is contradicted. n is not of the form 5k + 1
b) Suppose n = 5k + 2
   then $n^2 = (5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$
   which is of the form 5t + 4 for another integer t
   Thus n^2 is not a multiple of 5 which is false and the new assumption is contradicted. n is not of the form 5k + 2
c) Suppose n = 5k + 3
   Then $n^2 = (5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$ which is of the form 5t + 4 for another integer t. Thus n^2 is not a multiple of 5 which is false and the new assumption is contradicted. n is not of the form 5k + 3
d) Suppose n = 5k + 4
   Then $n^2 = (5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1$ which is of the form 5t + 1 for another integer t. Thus n^2 is not a multiple of 5 which is false and the new assumption is contradicted. n is not of the form 5k + 4

The assumption that n is not a multiple of five is contradicted. The conjecture is therefore true and for any integer n, if n^2 is a multiple of 5 then n itself is a multiple of 5

Q16: Assume that the statement is false: that is, assume that there is a largest integer say, P

Since 2 is a positive integer, $\mathbf{P \geq 2}$

But P^2 is a positive integer, thus $P \geq P^2$

Divide by P to give $\mathbf{1 \geq P}$

But $P \geq 2$ and $1 \geq P$ cannot both be true. The statements contradict.

Therefore the assumption that P is the largest integer is false.

The original conjecture that there is no largest positive integer must then be true.

Q17: Assume that the conjecture is false. Assume that $nx < y$ for every choice of positive integer n

Thus $n < {}^{y}/_{x}$

$\Rightarrow$ all positive integers are bounded above by ${}^{y}/_{x}$

$\Rightarrow$a largest integer exists.

By the previous question 'There is no largest positive integer.' The assumption is contradicted and the original conjecture is true.

Q18: Assume that $\sqrt{2}$ is rational.

Therefore $\sqrt{2} = \frac{p}{q}$ where p and q are positive integers with no common factors. (Otherwise the fraction could be reduced.)

So $2q^2 = p^2 \Rightarrow p^2$ is even.

Now if p is odd, then p^2 is odd.

But here p^2 is even: so **p is even**

and $p = 2m$ for some integer m

Thus $2q^2 = 4m^2 \Rightarrow q^2 = 2m^2$

Using the same argument again

$q^2 = 2m^2 \Rightarrow q^2$ is even and so **q is even**.

Thus both p and q are even but both were assumed to have no common factors. This is a contradiction and the assumption that $\sqrt{2}$ is rational is false.

The original conjecture that $\sqrt{2}$ is not a rational number is therefore true.

Q19: Suppose that the conjecture is untrue and assume that there are a finite number of primes.

So there is a collection of all primes $P_1, P_2, P_3, ..., P_k$

Let $N = (P_1 \times P_2 \times ... \times P_k) + 1$

Then N is obviously larger than all P_k

Since $P_1, ..., P_k$ is the collection of all primes then N is composite.

If N is composite then N has a prime divisor Q

But since all the primes $P_1, P_2, P_3, ..., P_k$ leave remainder 1 when dividing N, none of these is equal to Q

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This is impossible.

The assumption that there is a finite number of primes is false.

The conjecture that there is an infinite number of primes is true.

Induction exercise (page 176)

Q20: Check when r = 1

Then $1 = \frac{1(1+1)(2+1)}{6} = 1$. The result is true for r = 1

Assume true for r = k

Then $1 + 4 + 9 + ... + (k - 1)^2 + k^2 = \frac{k(k+1)(2k+1)}{6}$

Consider r = k + 1

Then $1 + 4 + 9 + ... + (k - 1)^2 + k^2 + (k + 1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

So the result is true for r = k + 1 if it is true for r = k. But it is true for r = 1 and so by the principal of mathematical induction the conjecture is true for all $r = n \in \mathbb{N}$

Q21: Check for n = 3

Then $3^2 - 2 \times 3 - 1 = 2$ and $2 > 0$

The result holds for n = 3

Assume that the conjecture is true for n = k

Then $k^2 - 2k - 1 > 0$

Consider n = k + 1

Then $(k + 1)^2 - 2(k + 1) - 1 = (k^2 - 2k - 1) + 2k - 1$

But $(k^2 - 2k - 1) + 2k - 1 > k^2 - 2k - 1 > 0$ since $n \geq 3$

Thus the result is true for n = k + 1 if it is true for n = k

But it is also true for n = 3 and so by the principle of mathematical induction the conjecture is true for all $n \geq 3$

Q22: A bit of number crunching will find that when m = 7 the result holds.

Then $7! > 3^7$

Assume the result is true for n = k then $k! > 3^k$

Consider n = k + 1

Then $(k + 1)! > (k + 1) \times 3^k > 3^{k+1}$since $k > 7$ from the basis step.

Thus the result holds for n = k + 1 if it is true for n = k but it is true for n = 7.

So by the principle of mathematical induction the conjecture is true for all $n > 7$

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Q23: The first step here is to find a conjecture for all positive integers which will be easier to prove.

Since n is a positive odd integer, let n = 2m + 1

Then $n^2 - 1$ becomes $(2m + 1)^2 - 1 = 4m^2 + 4m$ and the conjecture can then be restated as $4m^2 + 4m$ is divisible by 8 for all positive integers m

Check m = 1 which gives 8 and 8 is divisible by 8. It is true for m = 1

Assume that the conjecture is true for m = k then $4k^2 + 4k = 8a$ for some integer a

Consider m = k + 1

Thus $4(k + 1)^2 + 4(k + 1) = 4k^2 + 8k + 4 + 4k + 4 = 4k^2 + 12k + 8$

So the effect in letting m = k + 1 from m = k is to add 8k + 8

Thus $4(k + 1)^2 + 4(k + 1) = 8a + 8k + 8 = 8(a + k + 1)$

That is $4(k + 1)^2 + 4(k + 1)$ is divisible by 8

Thus the result holds for m = k + 1 if it is true for m = k

But it is true for m = 1 and so by the principle of mathematical induction the new conjecture is true for all m positive integers.

But n = 2m + 1 and so the original conjecture that $n^2 - 1$ is divisible by 8 for all positive odd integers is now true.

Note that the new conjecture of $4m^2 + 4m$ is divisible by 8 for all positive integers m could have been simplified and stated as $m^2 + m$ is divisible by 2 for all positive integers m.

Q24: Check for n = 10

Then $10^{10} = 10^{10}$

The result is true for n = 10

Assume that the result is true for n = k

Then $10^k \geq k^{10}$

Consider n = k + 1

Then $10^{k+1} \geq 10 \times k^{10} \geq (k + 1)^{10}$ (check using the binomial expansion).

So the result is true for n = k + 1 if it is true for n = k

But it is true for n = 1 and so by the principle of mathematical induction the conjecture is true for all $n \in \mathbb{N}$

Contrapositive exercise (page 178)

Q25: The contrapositive states that if n is odd then n^3 is odd.

So n = 2k + 1 for some k

Thus $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^2 + 6\ k^2 + 3k) + 1$

That is, if n is odd then n^3 is odd and the original conjecture is proved.

Q26: The contrapositive implication is if n can be expressed as a sum of two square integers then n does not have remainder 3 on division by 4.

If n does not have remainder 3 on division by 4 then n takes the form 4k, 4k + 1 or 4k + 2. This proof now involves four cases (proof by exhaustion).

Let $n = x^2 + y^2$ where x and y are integers. Then x and y are either even or odd.

a) Consider when x is even and y is even
   then x = 2m and y = 2l for some integers m and l
   Then $n = x^2 + y^2 = 4m^2 + 4l^2 = 4(m^2 + l^2)$ and n has the form 4k
b) When x is even and y is odd
   then x = 2m and y = 2l + 1 for some integers m and l
   Then $n = 4m^2 + (2l + 1)^2$
   $= 4m^2 + 4l^2 + 4l + 1$
   $= 4(m^2 + l^2 + l) + 1$ and n has the form 4k + 1
c) When x is odd and y is even the same result as for part 2 is found.
d) When x is odd and y is odd
   then x = 2m + 1 and y = 2l + 1 for some integers m and l
   Then $n = (2m + 1)^2 + (2l + 1)^2$
   $= 4m^2 + 4m + 1 + 4l^2 + 4l + 1$
   $= 4(m^2 + m + l^2 + l) + 2$ and n has the form 4k + 2

The contrapositive conjecture and so the original conjecture is proved.

Q27: Using the contrapositive which states

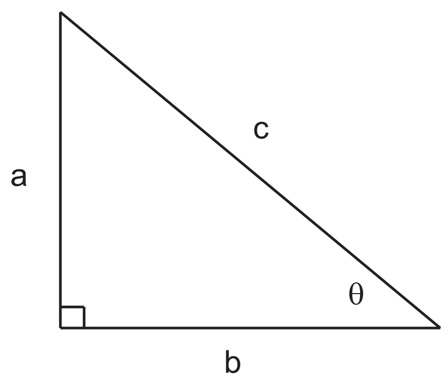
if n is even then n^2 is not of the form 4m + 1, there are two cases to consider.

- When n = 4m then $n^2 = 16m^2 = 4(4m^2)$ and is not of the form 4m + 1
- When n = 4m + 2 then $n^2 = 16m^2 + 16m + 4 = 4(4m^2 + 4m + 1)$ which is not of the form 4m + 1

The contrapositive holds and the original conjecture is true.

Direct proof exercise (page 180)

Q28: Sketch the triangle



$\cos\theta = \frac{b}{c}$ and $\sin\theta = \frac{a}{c}$

so $\cos^2\theta + \sin^2\theta = \frac{b^2}{c^2} + \frac{a^2}{c^2}$

But by Pythagoras, $a^2 + b^2 = c^2$

so $\frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{c^2}{c^2} = 1$

Q29: Using the combination rules and known results

$$\begin{aligned}\sum_{r=1}^{n}(2r - 1) &= 2\sum_{r=1}^{n} r - n\\ &= \frac{2n(n+1)}{2} - n\\ &= n^2 + n - n\\ &= n^2\end{aligned}$$

Q30: $b > c \Rightarrow b - c > c - c = 0$

$\Rightarrow a(b - c) > 0$

$\Rightarrow ab - ac > 0$

$\Rightarrow ab > ac$

Q31: $r(r + 1)(r + 2) = r^3 + 3r^2 + 2r$ so

$$\begin{aligned}\sum_{r=1}^{n} r(r+1)(r+2) &= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r\\ &= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + \frac{2n(n+1)}{2}\\ &= \frac{n(n+1)(n^2 + n + 4n + 2 + 4)}{4}\\ &= \frac{n(n+1)(n+2)(n+3)}{4}\end{aligned}$$

Division algorithm exercise (page 183)

Q32: The answers are

1. q = -9 and r = 3: -42 = -9 . 5 + 3
2. q = -3 and r = 4: 25 = -3 . -7 + 4
3. q = -1 and r = 5: 14 = -1 . -9 + 5

Q33: The answers are

1. q = -1 and r = 2: -1 = -1 . 3 + 2
2. q = 0 and r = 4: 4 = 0 . 9 + 4
3. q = - 4 and r = 0: -12 = -4 . 3 + 0

Answers from page 184.

Q34: All the possible forms of a number divided by 8 are

8k, 8k + 1, 8k + 2, 8k + 3, 8k + 4, 8k + 5, 8k + 6 and 8k + 7

Of these the even forms are 8k, 8k + 2, 8k + 4, 8k + 6 as each of these is of the form 2m for some integer m

Number bases exercise (page 186)

Q35: The binary equivalents are :

a) 101001100
   332 = 166 . 2 + 0
   166 = 83 . 2 + 0
   83 = 41 . 2 + 1
   41 = 20 . 2 + 1
   20 = 10 . 2 + 0
   10 = 5 . 2 + 0
   5 = 2 . 2 + 1
   2 = 1 . 2 + 0
   1 = 0 . 2 + 1
b) 111110101
c) 101011
d) 111111

Q36: The octal forms are:

a) 533
   347 = 43 . 8 + 3
   43 = 5 . 8 + 3
   5 = 0 . 8 + 5
b) 5554
c) 5704
d) 1026

Q37: The hexadecimal answers are :

a) FAE
   4014 = 250 . 16 + 14 (E)
   250 = 15 . 16 + 10 (A)
   15 = 0 . 16 + 15 (F)
b) 16C
c) 883
d) 13AA

Answers from page 187.

Q38: The factors of 2210 are 1, 2, 5, 13, 17, 2210 and the factors of 399 are 1, 3, 7, 19, 399

The only common factor is 1, thus gcd (2210, 399) = 1 and by definition the integers are relatively prime.

Euclidean algorithm exercise (page 189)

Q39: gcd (299, 1365) is 13

1365 = 4 . 299 + 169

299 = 1 . 169 + 130

169 = 1 . 130 + 39

130 = 3 . 39 + 13

39 = 3 . 13 + 0

Q40: The highest common factor is the same as the gcd.

The gcd (5187, 760) = 19

Q41: For coprime integers the gcd is 1

10465 = 1 . 5643 + 4822

5643 = 1 . 4822 + 821

4822 = 5 . 821 + 717

821 = 1 . 717 + 104

717 = 6 . 104 + 93

104 = 1 . 93 + 11

93 = 8 . 11 + 5

11 = 2 . 5 + 1

5 = 5 . 1 + 0

This gcd is 1 and the numbers have no common factor other than 1. They are coprime.

Q42:

5184 = 1 . 3024 + 2160

3024 = 1 . 2160 + 864

2160 = 2 . 864 + 432

864 = 2 . 432 + 0 and the gcd (3024, 5184) = 432

Q43: By the Euclidean algorithm the numerator and denominator have a gcd of 124. Dividing top and bottom by this gives a simplified fraction of $\frac{3}{35}$

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Answers from page 190.

Q44:

610 = 76 . 8 + 2

8 = 4 . 2 + 0

The gcd (610, 8) is 2

610 = 11 . 55 + 5

55 = 11 . 5 + 0

The gcd (610, 55) is 5

Linear combinations exercise (page 193)

Q45:

693 = 8 . 84 + 21

84 = 4 . 21

The gcd is 21

working backwards 21 = 693 . 1 - 84 . 8 and x = 1, y = -8

Q46:

10080 = 2 . 3705 + 2670

3705 = 1 . 2670 + 1035 (line a)

2670 = 2 . 1035 + 600 (line b)

1035 = 1 . 600 + 435 (c)

600 = 1 . 435 + 165 (d)

435 = 2. 165 + 105 (e)

165 = 1 . 105 + 60 (f)

105 = 1 . 60 + 45 (g)

60 = 1 . 45 + 15 (h)

45 = 3 . 15 + 0 (i)

The gcd is 15

Working backwards gives 15 = 60 - 1 . 45 using line (h)

15 = 60 - 1(105 - 1. 60) = - 1 . 105 + 2 . 60 using line (g)

15 = -1 . 105 + 2(165 - 1 . 105) = 2 . 165 - 3 . 105 using line (f)

15 = 2 . 165 - 3(435 - 2 . 165) = -3 . 435 + 8 . 165 using line (e)

15 = -3 . 435 + 8(600 - 1 . 435) = 8 . 600 - 11 . 435 using line (d)

15 = 8 . 600 - 11(1035 - 1 . 600) = -11 . 1035 + 19 . 600 using line (c)

15 = -11 . 1035 + 19(2670 - 2 . 1035) = 19 . 2670 - 49 . 1035 using line (b)

15 = 19 . 2670 - 49(3705 - 1 . 2670) = -49 . 3705 + 68 . 2670 using line (a)

15 = -49 . 3705 + 68(10080 - 2 . 3670) = 68 . 10080 - 185 . 3705

x = 68 and y = -185

Q47: 12 = 336 . 7 - 180 . 13 so x = 7 and y = -13

Q48: x = -7 and y = 18

Q49:

585 = 5 . 104 + 65

104 = 1 . 65 + 39

65 = 1 . 39 + 26

39 = 1 . 26 + 13

26 = 2 . 13 + 0

The gcd is 13

Working backwards eliminating the remainders to solve for x and y gives

13 = 39 - 1 . 26

13 = 39 - 1(65 - 1 . 39) = -1 . 65 + 2 . 39

13 = -1 . 65 + 2(104 - 1 . 65) = 2 . 104 - 3 . 65

13 = 2 . 104 - 3(585 - 5 . 104) = -3 . 585 + 17 . 104

Thus x = -3 and y = 17

For the general solution x = -3 + (104/13)m = -3 + 8m

and y = 17 - (585/13)m = 17 - 45m

The general solution is 13 = 585(-3 + 8m) + 104(17 - 45m)

Q50:

204 = 3 . 56 + 36

56 = 1 . 36 + 20

36 = 1 . 20 + 16

20 = 1 . 16 + 4

16 = 4 . 4 + 0

The gcd is 4 and 4 $|$ 20 so there is a solution.

Thus working backwards

4 = 20 - 1. 16

4 = 20 - 1(36 - 1 . 20) = -1 . 36 + 2 . 20

4 = -1 . 36 + 2(56 - 1 . 36) = 2 . 56 - 3 . 36

4 = 2 . 56 - 3(204 - 3 . 56) = -3 . 204 + 11 . 56

Multiply by 5 since 5 $\times$ 4 = 20

20 = -15 . 204 + 55 . 56

The general solution is given by x = -15 + ($^{56}/_{4}$)m and y = 55 - ($^{204}/_{4}$)m

That is, x = -15 + 14m and y = 55 - 51m

The general solution is 20 = 204(-15 + 14m) + 56(55 - 51m)

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Review exercise (page 196)

Q51: Check when r = 1 then 1 = $\frac{1(2)^2}{4}$ = 1 as required.

Assume that it is true for n = k

Then $1 + 2^3 + 3^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$

Consider n = k + 1 then

$1 + 2^3 + 3^3 + ... + k^3 + (k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^3$

$= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4}$

$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$

$= \frac{(k+1)^2(k+2)^2}{4}$

$= \frac{(k+1)^2((k+1)+1)^2}{4}$

The result is true for n = k + 1 if it is true for n = k. But it is also true for n = 1 and so by the principle of mathematical induction it is true for all n

Q52:

943 = 1 . 779 + 164

779 = 4 . 164 + 123

164 = 1 . 123 + 41

123 = 3 . 41 + 0

The greatest common divisor is 41

Q53: Let the integers be n, n +1 and n + 2

The conjecture states that $n^3 + (n+1)^3 + (n+2)^3 = 9a$ for some integer a

Check when n = 1

$1^3 + 2^3 + 3^3 = 36$ and $36 = 9 \times 4$

The result is true for n = 1

Assume that the result is true for n = k

Then $k^3 + (k+1)^3 + (k+2)^3 = 9b$ for some integer b

Consider n = k + 1 then the effect is the difference between $k^3 + (k+1)^3 + (k+2)^3$ and $(k+1)^3 + (k+2)^3 + (k+3)^3$ which is $9k^2 + 27k + 27$

Thus when n = k + 1

$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9b + 9k^2 + 27k + 27 = 9(b + k^2 + 3k + 3)$. That is, it is divisible by 9

The result is true for n = k + 1 if it is true for n = k

But it is also true for n = 1 and so by the principle of mathematical induction the conjecture is true for all n

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Q54:

5141 = 1 . 3763 + 1378

3763 = 2 . 1378 + 1007

1378 = 1 . 1007 + 371

1007 = 2 . 371 + 265

371 = 1 . 265 + 106

265 = 2 . 106 + 53

106 = 2 . 53 + 0

The greatest common divisor is 53

Advanced review exercise (page 196)

Q55:

247 = 1 . 139 + 108

139 = 1 . 108 + 31

108 = 3 . 31 + 15

31 = 2 . 15 + 1

15 = 15 . 1 + 0

So working backwards

1 = 31 - 2 . 15

= 31 - 2(108 - 3 . 31) = - 2 . 108 + 7 . 31

= -2 . 108 + 7(139 - 1 . 108) = 7 . 139 - 9 . 108

= 7 . 139 - 9(247 - 1 . 139) = -9 . 247 + 16 . 139

so 1 = 247x + 139y where x = -9 and y = 16

Q56:

252 = 1 . 160 + 92

160 = 1 . 92 + 68

92 = 1 . 68 + 24

68 = 2 . 24 + 20

24 = 1 . 20 + 4

20 = 5 . 4 + 0

so working backwards gives

4 = 24 - 1 . 20

4 = 24 - 1(68 - 2 . 24) = -1 . 68 + 3 . 24

4 = -1 . 68 + 3(92 - 1 . 68) = 3 . 92 - 4 . 68

4 = 3 . 92 - 4(160 - 1 . 92) = -4 . 160 + 7. 92

4 = -4 . 162 + 7(252 - 1 . 160) = 7 . 252 - 11 . 160

So 41 = 252x + 160y where x = 7 and y = -11

Q57:

297 = 1 . 180 + 117

180 = 1 . 117 + 63

117 = 1 . 63 + 54

63 = 1 . 54 + 9

54 = 6 . 9 + 0

So gcd (297, 180) = 9

Working backwards gives

9 = 63 - 1 . 54

9 = 63 - 1(117 - 1 . 63) = -1 . 117 + 2 . 63

9 = -1 . 117 + 2(180 - 1 . 117) = 2 . 180 - 3 . 117

9 = 2 . 180 - 3(297 - 1 . 180) = -3 . 297 + 5 . 180

so 9 = 297x + 180y where x = -3 and y = 5

Set review exercise (page 197)

Q58: The answer is only available on the web.

Q59: The answer is only available on the web.

Q60: The answer is only available on the web.

Q61: The answer is only available on the web.

Well done! This is the end of the final exercise in the last topic in this Advanced Higher in Maths. We hope that you have found the course worthwhile and interesting.

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