Advanced Higher Mathematics

Sequences And Series

EXERCISE 2A

1 Identify a and d in each of the following arithmetic sequences.

a 3, 5, 7, ... **b** 4, 5, 6, ... **c** 3, 1, −1, ...

d −2, 2, 6, ... **e** −2, −5, −8, ... **f** 0, 3, 6, ...

g 2, 1.9, 1.8, ... **h** $\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \ldots$ **i** $\frac{1}{8}, \frac{3}{16}, \frac{1}{4}, \ldots$

2 Find the nth term for each of these arithmetic sequences.

a 4, 7, 10, ... **b** 8, 5, 2, ... **c** 9, 5, 1, ...

d −3, 2, 7, ... **e** 17, 11, 5, ... **f** $\frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \ldots$

g $\frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \ldots$ **h** $\frac{8}{9}, \frac{2}{3}, \frac{4}{9}, \ldots$ **i** −0.6, −1.3, −2, ...

3 **a** Find the value of n when $a = -3$, $d = 2$ and $u_n = 15$.

b Which term in the arithmetic sequence 4, 1, −2, ... is −14?

c If, in an arithmetic sequence, $u_1 = 1$ and $u_2 = 1.5$, which term has the value 31?

4 **a** Find the value of d when $a = 8$ and $u_{12} = 41$.

b An arithmetic sequence starts with 3. Its 100th term is −393. What is the common difference between terms?

c The first term of an arithmetic sequence is 1. List the first four terms if the 10th term is 4.

5 **a** For a particular arithmetic sequence, $u_{20} = 65$. If $d = 3$, find a.

b An arithmetic sequence with 12 as the common difference between terms has 231 for its 19th term. How does the sequence start?

c Counting *down* in twos, where do you start, to finish on 4 after 23 terms?

6 **a** Identify the arithmetic sequence in each case by quoting the first four terms.

(i) $u_5 = 24$ and $u_{10} = 49$ **(ii)** $u_8 = 11$ and $u_{15} = -3$ **(iii)** $u_9 = 7$ and $u_{17} = 13$

b An arithmetic sequence has 106 as its 14th term and 64 for its 8th term.

(i) Identify the sequence.

(ii) Which number between 150 and 160 is a term of the sequence?

7 An arithmetic sequence is defined by $u_n = 2n + 6$.

a Find analytically the value of x such that $2u_x = u_{3x}$.

b If a sequence is defined by $u_n = pn + q$ show that if $2u_x = u_{3x}$ then p is a factor of q.

8 By considering u_x and u_{x+1} show that a sequence defined by $u_n = pn + q$, where p and q are constants, is an arithmetic sequence.

Sum to n terms of an arithmetic series

EXERCISE 3A

1 **a** Calculate the sum to 10 terms of the arithmetic series which starts $3 + 10 + 17 + \cdots$

b Find S_{12} for an arithmetic series when $u_1 = 8$, $u_2 = 27$ and $u_3 = 46$.

c Find the required sum when each of the following is an arithmetic series.

(i) $3 + 5 + 7 + \cdots$: S_{25} (ii) $7 + 11 + 15 + \cdots$: S_{100}

(iii) $(-2) + (-10) + (-18) + \cdots$: S_{16} (iv) $-5 - 3 - 1 - \cdots$: S_7

2 The first two terms of an arithmetic sequence are 9 and 12 in that order.

a Find the sum of the first (i) 18 terms, (ii) 19 terms.

b Hence calculate the 19th term.

c Repeat this process if the first two terms are 12 and 9 in that order.

3 Find the following sums, given that each is an arithmetic series.

a $2 + 3 + 4 + \cdots + 25$

b $12 + 19 + 26 + \cdots + 355$

c $-5 + (-7) + \cdots + (-43)$

d $8 - 2 - 12 - \cdots - 62$

e $0.01 + 0.32 + 0.63 + \cdots + 2.8$

f $\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + \cdots + 4$

4 **a** The sum of the first 50 terms of an arithmetic series is 2800. The common difference is 2. What is the first term?

b The sum of the first 10 terms of an arithmetic series is 25. The common difference is 0.3. What is the first term?

5 **a** The first term of an arithmetic progression is 6 and the last term is 78. The sum of the terms is 798. How many terms are in the progression?

b List the first four terms of an arithmetic progression if the last term is ten times the first term and the sum of the terms is 121. [There are two possible solutions.]

6 **a** An arithmetic series starts with 3, has 25 terms and totals 2175. What is the common difference?

b What is the common difference of an arithmetic series which has 100 terms, the first term being 0.5, and which totals 545?

7 **a** The first three terms of an arithmetic sequence total 24. The next three total 69. What is the sum of the three after that?

b The sum of the first four terms of an arithmetic sequence is −36. The next five total −135. What is the 21st term?

8 **a** Given that $u_{14} = 20$ and $S_{15} = 165$, calculate u_{16} and S_{17}.

b The first term of a sequence is 10. The sum to 22 terms equals the 22nd term.

(i) What is the common difference? (ii) What is the 22nd term?

9 A historical problem (sixteenth century).
100 eggs are placed in a straight line, one step apart. A basket is situated one step from the start of the line. A person walks away from the basket, picks the first egg up and returns it to the basket. He then walks to the second egg, picks it up and returns it to the basket. He continues in this fashion until all the eggs are in the basket. How many steps did he take?

Geometric sequences

EXERCISE 4A

1 For each of the following geometric sequences, **(i)** identify a and r, **(ii)** hence find an expression for the nth term.

a 1, 3, 9, 27, ...
b 4, −8, 16, −32, ...
c 1458, 486, 162, 54, ...
d 3072, −768, 192, −48, ...
e 7, 0.7, 0.07, 0.007, ...
f $\frac{2}{3}, \frac{4}{15}, \frac{8}{75}, \frac{16}{375}, \ldots$
g 0.16, 0.128, 0.1024, ...
h 23.2, 4.64, 0.928, ...

2 **a** The first term of a geometric sequence is 2 and the common ratio is 6. What is the fifth term?
b If the first term of a geometric sequence is 0.8 and the second term is 0.24, calculate the sixth term.

3 The sixth term of a geometric sequence is 31 250 and the tenth term is 19 531 250.
a Find an expression for the nth term.
b What is the difference between the eighth and ninth term?

4 **a** The first two terms of a geometric sequence are 1024 and 2560 in that order. Which term has a value of 625 000?
b If the initial two terms had been $u_1 = 2560$ and $u_2 = 1024$, determine the first term that has a value less than 1.

5 **a** The first term of a geometric sequence is 25. The tenth term is 139. Calculate the common ratio to two decimal places.
b The common ratio of a geometric sequence is 0.98. The eighth term is 131. What is the first term to the nearest whole number?

6 **a** Show that a, $a + d$, $a + 2d$ are not the first three terms of a geometric sequence.
b If a, $a + d$, $a + 6d$ are the first three terms of a geometric sequence, **(i)** express a in terms of d, **(ii)** find the common ratio.
c If a, $a + d$, $a + xd$ are the first three terms of a geometric sequence express the common ratio in terms of x only.

7 **a** Show that $2 \sin x$, $\sin 2x$ and $\sin 2x \cos x$ could be the first three terms of a geometric sequence.
b If $x = \frac{\pi}{4}$ radians express the first ten terms in their simplest form.

8 **a** If p, x, q are the first three terms of a geometric sequence then x is called the geometric mean between p and q.
Find the geometric mean between **(i)** 6 and 24 **(ii)** $\frac{3}{4}$ and $\frac{27}{25}$.
b 1250, x, y, 10 form a geometric sequence. Calculate x and y.

EXERCISE 5A

1 Sum each of the following geometric sequences to the required number of terms.

a 2, 6, 18, … to 7 terms

b 4, 8, 16, … to 9 terms

c 3, −15, 75, … to 8 terms

d 4, 20, 100, … to 6 terms

e 1, −2, 4, … to 10 terms

f 2, −8, 32, … to 7 terms

2 Calculate each of these geometric series to the required number of terms.

a $\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \cdots$ to 6 terms

b $\frac{4}{3} + \frac{1}{2} + \frac{3}{16} + \cdots$ to 8 terms

c $12 - 10 + \frac{25}{3} - \cdots$ to 7 terms

d $\frac{1}{4} + \frac{2}{5} + \frac{16}{25} + \cdots$ to 9 terms

e $4 - \frac{1}{3} + \frac{1}{36} - \cdots$ to 10 terms

f $50 - 20 + 8 - \cdots$ to 8 terms

3 **a** How many terms must be added for the geometric series $9 + 18 + 36 + \cdots$ to equal 9207?

b How many terms must be added for the geometric series $9 + 6 + 4 + \cdots$ to exceed 26?

4 A geometric series has a common ratio of 2. Its sum to 6 terms is 42.

a Calculate the first term.

b Find the sum to eight terms.

5 **a** For the geometric series with a first term a and a common ratio r show that

$$u_n + u_{n+1} = r^{n-1}(u_1 + u_2)$$

b The first two terms of a geometric series add up to 4. The fourth and fifth terms total 108. Identify the series.

c The first three terms of a geometric series total 7. The sixth, seventh and eighth total 224. List the first five terms of the series.

d The first three terms of a geometric series total 93. The first six terms of the series total 11 718.

(i) State the sum of the fourth, fifth and sixth terms.

(ii) Identify the series.

(iii) Compare this method with that of Example 3.

Sum to infinity of a geometric series

EXERCISE 6A

1 Find the sum to infinity of the following infinite geometric progressions.

a $1 + 0.5 + 0.25 + \cdots$ **b** $12 + 6 + 3 + \cdots$ **c** $7 + 1 + \frac{1}{7} + \cdots$

d $8 - 4 + 2 - \cdots$ **e** $81 - 9 + 1 - \cdots$ **f** $100 + 10 + 1 + \cdots$

2 Identify which of the following geometric series tend to a limit and find the limit.

a $18 + 3.6 + 0.72 + \cdots$ **b** $0.5 - 1 + 2 - \cdots$ **c** $-5 - 2.5 - 1.25 - \cdots$

d $0.1 - 0.2 + 0.4 - \cdots$ **e** $\frac{3}{5} + \frac{6}{15} + \frac{12}{45} \cdots$ **f** $\frac{1}{12} + \frac{1}{6} + \frac{1}{3} \cdots$

3 **a** A geometric series has a sum to infinity of 40. If the common ratio is 0.2, what is the first term of the series?

b The first term of an infinite geometric series is 18. The third term is 2.88.

(i) Show that the series has a limit.

(ii) Find the limit.

4 Express each of the following recurring decimals as an infinite geometric series, and hence as a vulgar fraction in its simplest form.

a 0.141 41 … **b** 0.144 44 … **c** 0.270 270 …

5 **a** By considering 0.014 14 … as $\frac{1}{10}$(0.141 414 …) express it as a vulgar fraction in its simplest form.

b By considering 0.614 14 … as $\frac{6}{10} + \frac{1}{10}$(0.141 414 …) express it as a vulgar fraction in its simplest form.

c Express each of the following recurring decimals as a vulgar fraction in its simplest form.

(i) 0.011 11 … **(ii)** 0.322 222 … **(iii)** 0.436 363 6 … **(iv)** 0.501 818 181 81 …

6 Given that 0.64 and 0.128 are two adjacent terms of an infinite geometric series with a sum to infinity of 20,

a find the first term,

b find the partial sum S_5.

7 A ball is thrown to the ground. It bounces to a height of 3 m. The characteristics of the rubber are such that, thereafter, the ball rebounds to 0.4 of its previous height at each bounce. Let u_n represent the distance travelled, from ground to ground, in the nth bounce ($u_1 = 6$ metres).

a Write down the first four terms of the geometric series generated.

b Work out the total distance travelled after

(i) five bounces,

(ii) ten bounces.

c What is the limit of this distance as n tends to infinity?

EXERCISE 8

1 Expand each of the following.

a $\sum_{r=1}^{5} r$ **b** $\sum_{r=1}^{4} r^2$ **c** $\sum_{r=0}^{3} r^3$ **d** $\sum_{r=0}^{4} r!$

e $\sum_{r=1}^{6} (-1)^r$ **f** $\sum_{r=1}^{5} (-1)^r r^2$ [What effect does the factor $(-1)^r$ have on the terms?]

g $\sum_{r=1}^{6} (-1)^{r+1}$ **h** $\sum_{r=1}^{5} (2r+3)$

2 Evaluate each of the following.

a $\sum_{r=1}^{5} (2r+3)$ **b** $\sum_{r=3}^{7} (3r+1)$ **c** $\sum_{r=4}^{8} (3-2r)$ **d** $\sum_{r=-1}^{3} (5-r)$

3 Express each of these series using sigma notation.

a $1 + x + x^2 + x^3 + x^4 + x^5$ **b** $1 - x + x^2 - x^3 + x^4 - x^5$

c $1 + 2x + 4x^2 + 8x^3 + 16x^4$ **d** $1 - 2 + 3 - 4 + 5 - 6$

4 Express each of these series **(i)** in terms of $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$, **(ii)** in terms of n.

a $\sum_{r=1}^{n} (3r+2)$ **b** $\sum_{r=1}^{n} (4r+1)$ **c** $\sum_{r=1}^{n} (5r-3)$ **d** $\sum_{r=1}^{n} (4-6r)$

e $\sum_{r=1}^{n} (5-2r)$

5 The arithmetic progression $13 + 19 + 25 + 31 + \cdots + 67$ has an nth term, $u_n = 6n + 7$; 67 is the 10th term and so the progression can be written as $\sum_{r=1}^{10}(6r + 7)$.

Express each of the following arithmetic progressions in sigma notation using a similar method.

a $12 + 16 + 20 + 24 + \cdots + 88$

b $18 + 28 + 38 + 48 + \cdots + 108$

c $25 + 20 + 15 + \cdots + (-50)$

d $19 + 16 + 13 + \cdots + (-11)$

e $1.6 + 1.7 + 1.8 + \cdots + 2.4$

f $\frac{9}{4} + \frac{5}{2} + \frac{11}{4} + \cdots + 22$

6 **a** Check that the progression $-13 + 19 - 25 + 31 - \cdots + 67$ can be written as

$$\sum_{r=1}^{10}(-1)^r(6r + 7)$$

b Use the same strategy to express each of the following in sigma notation.

(i) $-5 + 13 - 21 + 29 - \cdots - 85$

(ii) $-7 + 16 - 25 + 34 - \cdots + 106$

(iii) $-100 + 90 - 80 + \cdots + 10$

(iv) $-26 + 22 - 18 + \cdots - (-46)$

c Verify that $13 - 19 + 25 - 31 + \cdots - 67 = \sum_{r=1}^{10}(-1)^{r+1}(6r + 7)$ and hence express each of the following in sigma notation.

(i) $4 - 5 + 6 - 7 \cdots - 19$

(ii) $8 - 47 + 86 - 125 + \cdots - 437$

(iii) $285 - 268 + 251 - \cdots + 81$

(iv) $61 - 59 + 57 - 55 + \cdots + 29$

7 The geometric progression $3 + 6 + 12 + 24 + \cdots + 1536$ has an nth term, $u_n = 3 \times 2^{n-1}$; 1536 is its 10th term and so the progression can be written as $\sum_{r=1}^{10} 3 \times 2^{r-1}$.

Express each of the following geometric progressions in sigma notation using a similar method.

a $2 + 6 + 18 + 54 + \cdots + 4374$

b $3072 + 1536 + 768 + \cdots + 3$

c $2000 + 200 + 20 + \cdots + 0.002$

d $\frac{9}{4} + \frac{18}{12} + \frac{36}{36} + \cdots + \frac{576}{2916}$

Answers

CHAPTER 5

Exercise 1 (page 115)

1 **a** (i) 8 (ii) $u_n + 2$
b (i) 11 (ii) $u_n + 3$
c (i) −3 (ii) $u_n - 6$
d (i) 24 (ii) $2u_n$
e (i) 4 (ii) $\frac{u_n}{2}$
f (i) 22 (ii) $2u_n + 2$
g (i) 17 (ii) $2u_n - 1$
h (i) 106 (ii) $4u_n + 2$
i (i) 5 (ii) $\frac{u_n}{2} - \frac{3}{2}$

2 **a** $u_{n+1} = 3u_n + 2$
b (i) 17 (ii) 485

3 **a** (i) $r^2u_1 + rd + d$ (ii) $r^3u_1 + r^2d + rd + d$
b (i) 3 (ii) −2

4 **a** −2, unstable **b** 20, stable
c 0.5, unstable **d** 10, stable
e undefined **f** −10, stable

5 **a** 20 million litres
b stable, at 30 million litres

Exercise 2A (page 117)

1 **a** $a = 3, d = 2$ **b** 4, 1 **c** 3, −2
d −2, 4 **e** −2, −3 **f** 0, 3
g 2, −0.1 **h** $\frac{1}{12}, \frac{1}{12}$ **i** $\frac{1}{8}, \frac{1}{16}$

2 **a** $3n + 1$ **b** $-3n + 11$ **c** $-4n + 13$
d $5n - 8$ **e** $-6n + 23$ **f** $\frac{1}{8}n + \frac{3}{8}$
g $\frac{1}{10}n + \frac{1}{5}$ **h** $-\frac{2}{9}n + \frac{10}{9}$
i $-0.7n + 0.1$

3 **a** 10 **b** 7 **c** 61

4 **a** 3 **b** −4
c $1, 1\frac{1}{3}, 1\frac{2}{3}, 2$

5 **a** 8 **b** 15 **c** 48

6 **a** (i) 4, 9, 14, 19 (ii) 25, 23, 21, 19
(iii) $1, 1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}$
b (i) $a = 15, d = 7$ (ii) 155

7 **a** 3 **b** $x = \frac{q}{p}, x \in N$

8 $u_{x+1} - u_x = p$

9 **a** $\ln 6 - \ln 2 = \ln 18 - \ln 6 = \ln 3$
b $\ln(2 \times 3^{n-1})$ **c** 92

Exercise 2B (page 118)

1 **a** 1 **b** −11 **c** 29th

2 **a** 4, 8
b (i) proof (ii) 8, 8
c whenever the initial rectangle has one dimension equal to 2

3 **a** 17 **b** 5th

4 **a** 880
b potential common difference is not a whole number
5 common difference $= \pi$, $a = 20\pi$

Exercise 3A (page 120)

1 **a** 345 **b** 1350
c (i) 675 (ii) 20 500 (iii) −992 (iv) 7
2 **a** (i) 621 (ii) 684
b 63
c −243, −285, −42
3 **a** 324 **b** 9175 **c** −480
d −216 **e** 14.05 **f** 34
4 **a** 7 **b** 1.15
5 **a** 19
b 2, 3.8, 5.6, 7.4 and 1, $1\frac{3}{7}$, $1\frac{6}{7}$, $2\frac{2}{7}$
6 **a** 7 **b** 0.1
7 **a** 114 **b** −83
8 **a** 23, 212.5 **b** (i) −1 (ii) −11
9 10 100

Exercise 3B (page 121)

1 **a** $x = \dfrac{16}{(b-1)}$
b $(x, b) = (16, 2), (8, 3), (4, 5), (2, 9), (1, 17)$
c 520, 440, 400, 380, 370
2 **a** conditions lead to $x^3 - x = 3d \Rightarrow d = \dfrac{x(x-1)(x+1)}{3}$ and one of x, $(x-1)$ and $(x+1)$ is divisible by 3
b (i) $n^2 + n$, $4n^2 - n$, $10n^2 - 6n$, $20n^2 - 15n$
(ii) column 3 of triangle (see Bk 1 Ch 1)
3 **a** 4π, 4.02π, 4.04π, 4.06π, 4.08π
b 499π
4 **a** 7125 **b** £182 750
5 **a** 13 **b** 13

Exercise 4A (page 123)

1 **a** (i) 1, 3 (ii) $1 \times 3^{n-1}$
b (i) 4, −2 (ii) $4 \times (-2)^{n-1}$
c (i) 1458, $\frac{1}{3}$ (ii) $1458 \times \left(\frac{1}{3}\right)^{n-1}$
d (i) 3072, $-\frac{1}{4}$ (ii) $3072 \times \left(-\frac{1}{4}\right)^{n-1}$
e (i) 7, $\frac{1}{10}$ (ii) $7 \times \left(\frac{1}{10}\right)^{n-1}$
f (i) $\frac{2}{3}$, $\frac{2}{5}$ (ii) $\frac{2}{3} \times \left(\frac{2}{5}\right)^{n-1}$
g (i) 0.16, 0.8 (ii) $0.16 \times 0.8^{n-1}$
h (i) 23.2, 0.2 (ii) $23.2 \times 0.2^{n-1}$
2 **a** 2592 **b** 0.001 944
3 **a** $10 \times 5^{n-1}$ **b** 3 125 000
4 **a** 8 **b** 9
5 **a** 1.21 **b** 151
6 **a** d would need to be zero and hence $r = 1$
b (i) $a = \dfrac{d}{4}$ (ii) 5 **c** $x - 1$
7 **a** $\dfrac{\sin 2x}{2\sin x} = \dfrac{\sin 2x \cos x}{\sin 2x} = \cos x$
b $\sqrt{2}, 1, \dfrac{1}{\sqrt{2}}, \dfrac{1}{2}, \dfrac{1}{2\sqrt{2}}, \dfrac{1}{4}, \dfrac{1}{4\sqrt{2}}, \dfrac{1}{8}, \dfrac{1}{8\sqrt{2}}, \dfrac{1}{16}$
8 **a** (i) 12 (ii) $\dfrac{9}{10}$ **b** 250, 50

Exercise 4B (page 124)

1 **a** (i) 17 (ii) 27 (iii) 15
b (i) 17, 25 (ii) 27, 36 (iii) 17, 21
(iv) 18, 23
2 **a** (i) 2060 (ii) 2122 (iii) 2185
b $r = 1.03$ **c** 32
3 **a** $P\left(1 + \dfrac{r}{100}\right)$ **b** $P\left(1 + \dfrac{r}{100}\right)^2$
c $\left(1 + \dfrac{r}{100}\right)$ **d** $P\left(1 + \dfrac{r}{100}\right)^n$
4 **a** 7.7° **b** 46
5 **a** (i) $\dfrac{N_0e^{2a}}{N_0e^{a}} = \dfrac{N_0e^{3a}}{N_0e^{2a}}$ (ii) $r = e^a$
b 5
6 **a** (i) $\dfrac{V_0e^{\frac{-T}{5}}}{V_0e^{\frac{-T}{10}}} = \dfrac{V_0e^{\frac{-3T}{10}}}{V_0e^{\frac{-T}{5}}}$ (ii) $r = e^{\frac{-T}{10}}$
b (i) 6.9 years (ii) 23 years

Exercise 5A (page 127)

1 **a** 2186 **b** 2044 **c** −195 312
d 15 624 **e** −341 **f** 6554
2 **a** $\frac{91}{243}$ **b** 2.13 **c** 8.37
d 28.2 **e** 3.69 **f** 35.7
3 **a** 10 **b** 9
4 **a** $\frac{2}{3}$ **b** 170
5 **a** proof **b** 1, 3, 9, 27, 81
c 1, 2, 4, 8, 16
d (i) 11 625 (ii) 3, 15, 75, 375
6 **a** $\frac{1}{2}$ **b** 14 762
7 **a** $a = 1$, $r = \dfrac{1 \pm \sqrt{5}}{2}$ **b** (i) $\dfrac{1 - \phi^n}{1 - \phi}$
8 **a** (i) 1023, 1 048 575, 1 073 741 823
(ii) 1.25, 1.25, 1.25 (2 d.p.)
(iii) −682, −699 050, −715 827 882
(iv) 1.429, 1.429, 1.429 (3 d.p.)
(v) 130.26, 175.68, 191.52 (2 d.p.)
(vi) 571.62, 587.77, 588.22
b $|r| < 1$ each sequence tends to limit

Exercise 5B (page 128)

1 **a** 4.38 cm **b** 53.6 cm
2 **a** £188 355, £194 005, £199 825, £205 820, £211 995
b assume 1st yr constant £188 355 then increase is £20 191

3 a $57.6°$ **b** 10.8 seconds

4 a (i) 2 m (ii) $2\frac{11}{12}$ m (iii) 6.5 m

b (i) 9.75 m (ii) 1.5 times as big

5 a annual rate is 1.3%; $V_1 = 0.99V_0$; $V_2 = 0.97V_0$; $V_3 = 0.96V_0$; $V_4 = 0.95V_0$;

b 15% (nearest whole no.)

6 a (i) a (ii) $ra + d$ (iii) $r^2a + rd + d$ (iv) $r^3a + r^2d + rd + d$

b $u_n = ar^{n-1} + d\left(\frac{1 - r^{n-1}}{1 - r}\right)$

c $a\left(\frac{1 - r^n}{1 - r}\right) + d\left[\left(\frac{n - 1}{1 - r}\right) - \frac{r(1 - r^{n-1})}{(1 - r)^2}\right]$

Exercise 6A (page 131)

1 a 2 **b** 24 **c** $\frac{49}{6}$ **d** $\frac{16}{3}$ **e** 72.9 **f** $\frac{1000}{9}$

2 a 22.5 **c** -10 **e** $\frac{9}{5}$

3 a 32 **b** $r = \pm 0.4, \frac{90}{7}$ or 30

4 a $\frac{14}{100} + \frac{14}{10\,000} + \frac{14}{1\,000\,000} + \cdots = \frac{14}{99}$

b $\frac{1}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10\,000} + \cdots = \frac{1}{10} + \frac{4}{90} = \frac{13}{90}$

c $\frac{270}{1000} + \frac{270}{1\,000\,000} + \cdots = \frac{270}{999} = \frac{10}{37}$

5 a $\frac{14}{990} = \frac{7}{495}$ **b** $\frac{608}{990} = \frac{304}{495}$

c (i) $\frac{1}{90}$ (ii) $\frac{29}{90}$ (iii) $\frac{432}{990}$ (iv) $\frac{4968}{9900}$

6 a 16 **b** 19.9936

7 a 6, 2.4, 0.96, 0.384

b (i) 9.8976 m (ii) 9.998 95 m

c 10 m

Exercise 6B (page 132)

1 a (i) $1010\frac{10}{99}$

(ii) Achilles and the tortoise are at the same spot $1010\frac{10}{99}$ m from the start

b (i) 100, 1, 0.01, 0.0001 (ii) $101\frac{1}{99}$ s

(iii) Achilles won't overtake the tortoise before $101\frac{1}{99}$ seconds are up.

2 a (i) 12 times (ii) 15 min (iii) $\frac{15}{12}$ min

(iv) $a = 15$, $r = \frac{1}{12}$; sum $= 16\frac{4}{11}$ min after 3

b $32\frac{8}{11}$ min after 6

Exercise 7A (page 134)

1 a (i) 12 (ii) 4 (iii) 94

b $1 + 2x + 3x^2 + 4x^3 + \cdots$; No

c $1 + 3x + 6x^2 + 10x^3 + \cdots$

2 $1 - x + x^2 - x^3 + \cdots$

3 a $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$; not geometric

b $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

c $2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$

4 a $1 + x^2 + x^4 + x^6 + \cdots$ **b** even **c** $\sum_{r=0}^{\infty} x^{2r}$

5 a $1 + (1 - x) + (1 - x)^2 + (1 - x)^3 + \cdots$

b $-(1 - x) - \frac{(1 - x)^2}{2} - \frac{(1 - x)^3}{3} - \frac{(1 - x)^4}{4} - \cdots$

Exercise 7B (page 134)

1 a $1 - x^2 + x^4 - x^6 + \cdots$

b $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

c (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

2 a $1 + 2x + 4x^2 + 8x^3 + \cdots$; $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

b $1 - 3x + 9x^2 - 27x^3 + 81x^4 - \cdots$; $|x| < \frac{1}{3}$

c $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$; $|x| > 1$

3 a $1 + \sin x + \sin^2 x + \sin^3 x + \cdots$

b $1 + \sin^2 x + \sin^4 x + \sin^6 x + \cdots$

c $1 + \cos^2 x + \cos^4 x + \cos^6 x + \cdots$

4 a $\frac{1}{p} + \frac{q}{p^2} + \frac{q^2}{p^3} + \frac{q^3}{p^4} + \cdots$

b $\frac{1}{p} - \frac{q}{p^2} + \frac{q^2}{p^3} - \frac{q^3}{p^4} + \cdots$

5 a $\frac{1}{3} - \frac{4x}{9} + \frac{16x^2}{27} - \frac{64x^3}{81} + \cdots$; $-\frac{3}{4} < x < \frac{3}{4}$

b $\frac{1}{2} + \frac{3x}{4} + \frac{9x^2}{8} + \frac{27x^3}{16} + \cdots$; $-\frac{2}{3} < x < \frac{2}{3}$.

Each could have been expanded differently as our answer to **c** should illustrate:

c $\frac{1}{x - 2} = \frac{1}{x}.\frac{1}{1 - \frac{2}{x}} = \frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3} + \frac{8}{x^4} \cdots$;

$x > 2$ *or* $x < -2$ alternatively

$\frac{1}{x - 2} = \frac{1}{-2 + x} = \frac{1}{-2}.\frac{1}{1 - \frac{x}{2}}$

$= -\frac{1}{2} - \frac{x}{4} - \frac{x^2}{8} - \frac{x^3}{16} \cdots$; $-2 < x < 2$

6 a $\operatorname{cosec} x + \operatorname{cosec} x \cot x + \operatorname{cosec} x \cot^2 x + \operatorname{cosec} x \cot^3 x + \operatorname{cosec} x \cot^4 x \cdots$; $-1 < \cot x < 1$

b $\sec^2 x(1 + \tan^2 x + \tan^4 x + \tan^6 x + \tan^8 x + \tan^{10} x + \cdots)$ and since $\sec^2 x = 1 + \tan^2 x$, this expands as $(1 + \tan^2 x + \tan^4 x + \tan^6 x + \tan^8 x + \cdots) + (\tan^2 x + \tan^4 x + \tan^6 x + \tan^8 x + \cdots) = 1 + 2\tan^2 x + 2\tan^4 x + 2\tan^6 x + 2\tan^8 x + \cdots$