Outcome 5 – Systems of Linear Equations

Use matrix techniques to solve systems of linear equations.

Solve a 3 x 3 system of linear equations using Gaussian Elimination on an augmented matrix.

MATRICES

A **matrix (matrices)** is a rectangular array of numbers arranged in rows and columns, the array being enclosed in round (or square) brackets.

e.g. $\begin{pmatrix} x \\ y \end{pmatrix}$ 2 rows 1 column; $\begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$ 2 rows 2 columns; $\begin{pmatrix} 6 & 8 & 10 \\ 3 & 4 & 5 \end{pmatrix}$ 2 rows 3 columns; $\begin{pmatrix} 4 & -2 & 5 \end{pmatrix}$ 1 row 3 columns

Each number in the array is called an **entry** or an **element** of the matrix and is identified by first stating the row and then the column in which it appears.

A matrix is often denoted by a capital letter.

The **order** of a matrix is given by stating the number of rows followed by the number of columns.

eg. $A = \begin{pmatrix} 4 & 6 & 8 \\ 2 & 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 4 \\ 2 & -6 \end{pmatrix}$

A has 2 rows and 3 columns and is said to be of order 2 x 3 (read 2 by 3).

B has the same number of rows and columns and is called a **square** matrix **of order 2.**

In General , a matrix A, with m rows and n columns (order m x n), can be represented as follows, where a_{ij} denotes the row and column of each element.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & - & a_{2n} \\ a_{31} & a_{32} & a_{33} & - & - & - & a_{3n} \\ , & , & , & - & - & - & , \\ , & , & , & - & - & - & , \\ , & , & , & - & - & - & , \\ a_{m1} & a_{m2} & a_{m3} & - & - & - & a_{mn} \end{pmatrix}$$

The solution of a system of linear equations in three variables.

The general form of a system of equations is :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

or in matrix form $AX = B$

where $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

A is an 3 x 3 matrix , X is an 3 x 1 matrix and B is an 3 x 1 matrix.

Gaussian Elimination is best demonstrated by example.

The process is as follows:-
Solve :-

$$x + y + 2z = 3 \quad \ldots\ldots \text{row}_1$$
$$2x - y - z = 2 \quad \ldots\ldots \text{row}_2$$
$$3x - 2y + 2z = -2 \quad \ldots\ldots \text{row}_3$$

Eliminate x from row_1
ie. $\text{row}_2 - 2 \times \text{row}_1$

$$2x - y - z = 2$$
$$2x + 2y + 4z = 6 \quad \text{ie } -3y - 5z = -4 \quad \ldots\ldots \text{row}_4$$

Eliminate x from row_3
ie. $\text{row}_3 - 3 \times \text{row}_1$

$$3x - 2y + 2z = -2$$
$$3x + 3y + 6z = 9 \quad \text{ie } -5y - 4z = -11 \quad \ldots\ldots \text{row}_5$$

Eliminate y from row_5
ie. $\text{row}_5 - \frac{5}{3} \times \text{row}_4$

$$-5y - 4z = -11$$
$$-5y - \frac{25}{3}z = -\frac{20}{3} \quad \text{ie } \frac{13}{3}z = -\frac{13}{3}$$
$$z = -1$$

Substitute in row_4 $\quad -3y + 5 = -4 \quad y = 3$

Substitute in row_1 $\quad x + 3 - 2 = 3 \quad x = 2$

Notice that we arranged the system of equations until we arrived at

$$x + y + 2z = 3$$
$$-3y - 5z = -4$$
$$\frac{13}{3}z = -\frac{13}{3}$$

This working can be set out more neatly as shown below

x	y	z	b
1	1	2	3
2	−1	−1	2
3	−2	2	−2
0	−3	−5	−4
0	−5	−4	−11
0	0	$\frac{13}{3}$	$-\frac{13}{3}$

This whole process is called **Gaussian Elimination.**

For the following general 3 x 3 system

$$a_1x + b_1y + c_1z = d_1 \quad \ldots\ldots\ldots\ldots (1)$$
$$a_2x + b_2y + c_2z = d_2 \quad \ldots\ldots\ldots\ldots (2)$$
$$a_3x + b_3y + c_3z = d_3 \quad \ldots\ldots\ldots\ldots (3)$$

The 3 x 3 matrix of the coefficients of x, y and z,
$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$
is called the **augmented matrix.**

Leave (1) alone

Eliminate x from equation (2) by **subtracting**
m_1 $(= \frac{a_2}{a_1})$ **times** equation (1) **from** equation (2)
ie. in the above example $m_1 = 2$ giving $-3y - 5z = -4$ .... (4)

Similarly, eliminate x from equation (3) by **subtracting**
$m_2 = (\frac{a_3}{a_1})$ **times** equation (1) **from** equation (3)
ie. in the above example $m_2 = 3$ giving $-5y - 4z = -11$ .... (5)

Thus we have

$$\begin{array}{lcl} x + y + 2z = 3 & & \\ 2x - y - z = 2 & \longrightarrow & -3y - 5z = -4 \\ 3x - 2y + 2z = -2 & \longrightarrow & -5y - 4z = -11 \end{array}$$

Leave (4) alone

Eliminate y from equation (5) by **subtracting**
$m_3 = (\frac{5}{3})$ **times** equation (4) **from** equation (5)

ie. in the above example $m_3 = \frac{5}{3}$ giving $\frac{13}{3}z = -\frac{13}{3}$

cont'd ....

The last equation gives the value of z and we then back-substitute into the equation above to obtain the value of y and use both values to obtain x from the first equation.

This process is called **Gaussian Elimination.**
The final equations are called the **Pivotal Equations.**
m_1, m_2 etc are called the **Multipliers.**
The above example will look like this in tabular form.

m	x	y	z	b
1	1	1	2	3
2	2	−1	−1	2
3	3	−2	2	−2
1		−3	−5	−4
$\frac{5}{3}$		−5	−4	−11
			$\frac{13}{3}$	$-\frac{13}{3}$

Checks on your working can be added to the above table.
This column is the sum of the coefficients of x, y, z and b.
We then perform the same operations on this and check that the sum agrees.

m	x	y	z	b	*check*
1	1	1	2	3	7
2	2	−1	−1	2	2
3	3	−2	2	−2	1
1		−3	−5	−4	−12
$\frac{5}{3}$		−5	−4	−11	−20
			$\frac{13}{3}$	$-\frac{13}{3}$	0
check	6	−2	3		
from	(1 + 2 + 3)	(1 − 1 − 2)	(2 − 1 + 2)		

Note (i) You can **re-order** the set of original equations so that the smallest coefficient of x, y or z is in the first column before the first set of eliminations.

(ii) At the second set of eliminations you can reorder the 2 equations in the same way.

Example Use Gaussian Elimination to solve the following system of equations.

$$\begin{aligned} x + y + 2z &= 3 \quad \ldots\ldots\ldots\ R_1 \\ 4x + 2y + z &= 13 \quad \ldots\ldots\ldots\ R_2 \\ 2x + y - 2z &= 9 \quad \ldots\ldots\ldots\ R_3 \end{aligned}$$

	m	x	y	z	b	check	
	1	1	1	2	3	7	R_1
	4	4	2	1	13	20	R_2
	2	2	1	–2	9	10	R_3
$R_2 - 4R_1$			–2	–7	1	–8	R_4
$R_3 - 2R_1$			–1	–6	3	–4	R_5

Now change the order of the last 2 lines.

	m	x	y	z	b	check	
	1	1	1	2	3	7	R_1
	4	4	2	1	13	20	R_2
	2	2	1	–2	9	10	R_3 ▲
			▼ –1	–6	3	–4	R_5 ▼
			–2	–7	1	–8	R_4
$R_4 - 2R_5$				5	–5		R_6

From $5z = -5$ => $z = -1$

Substitute in $-y - 6z = 3$ => $y = 3$

Substitute in $x + y + 2z = 3$ => $x = 2$

Exercise 1

Solve the following equations using Gaussian Elimination (You may have to interchange rows before you start) :-

1. $$\begin{aligned} x + y + z &= 1 \\ 3x + 3y + z &= 4 \\ 3x + 2y + 2z &= 7 \end{aligned}$$

2. $$\begin{aligned} x - 2y + z &= 6 \\ 3x + y - 2z &= 4 \\ 7x - 6y - z &= 10 \end{aligned}$$

3. $$\begin{aligned} 5x - y + 2z &= 25 \\ 3x + 2y - 3z &= 16 \\ 2x - y + z &= 9 \end{aligned}$$

4. $$\begin{aligned} x + y + z &= 2 \\ 3x - y + 2z &= 4 \\ 2x + 3y + z &= 7 \end{aligned}$$

5. $$\begin{aligned} 5x - 3y + 6z &= 0 \\ x + 5y + 2z &= 0 \\ -x + 2y + 5z &= 0 \end{aligned}$$

Ill - conditioning

A system of equations is said to be ill-conditioned when **small changes** in the coefficients of x, y and z produce relatively **large changes** in the solution.

Therefore any uncertainty in the coefficients or round-off in the elimination could drastically affect our answers. This could occur if the coefficients result from experiment.

eg.
$$x + 1000y = 1$$
$$x + 999y = 2 \quad \text{has solutions } x = 1001 \text{ and } y = -1$$

However
$$x + 999y = 1$$
$$x + 1000y = 2 \quad \text{has solutions } x = -998 \text{ and } y = 1$$

A change of 1 in 1000 has brought drastic changes in our solutions.

One sign of ill-conditioning is that the pivots decrease in magnitude at each stage, tending towards zero.

Example Show, by calculating the exact values, that the following system demonstrates ill-conditioning.

$$1x + 1y = 3 \qquad 0{\cdot}1y = 1 \qquad \text{ie. } y = 10 \text{ and } x = -7$$
$$1x + 0{\cdot}9y = 2$$

$$1x + 0{\cdot}95y = 3 \qquad 0{\cdot}05y = 1 \qquad \text{ie } y = 20 \text{ and } x = 0.2$$
$$1x + 0{\cdot}9y = 2$$

This system is said to be ill-conditioned since relatively small changes in y produces a large change in the solution.

Exercise 2 Show, by calculating the exact values, that the following system demonstrates ill-conditioning.

(a) (i)
$$1x + 1y = 2$$
$$1x + 1{\cdot}000y = 2{\cdot}0001$$

(ii)
$$1x + 1y = 2$$
$$1x + 1{\cdot}0001y = 1{\cdot}9999$$

(b) (i)
$$1x + 0{\cdot}99y = 1.99$$
$$0{\cdot}99x + 0{\cdot}98y = 1.97$$

(ii)
$$1x + 0{\cdot}99y = 2{\cdot}00$$
$$0{\cdot}99x + 0{\cdot}98y = 1{\cdot}97$$

Answers

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1. $x = 5, y = -{}^{7}/_{2}, z = -{}^{1}/_{2}$.
2. $x = 5, y = 3, z = 7$.
3. $x = 5, y = 2, z = 1$.
4. $x = 3, y = 1, z = -2$.
5. $x = 0, y = 0, z = 0$.

Exercise 2 Page 79

1. (a) (i) $x = 1, y = 1$ (ii) $x = -1, y = 3$.

   (b) (i) $x = 0{\cdot}01, y = 2$ (ii) $x = -97, y = 100$

Gaussian elimination

Historical note

Gauss

This technique of solving a system of equations by expressing the system in augmented matrix form, reducing it to upper triangular form and then performing back-substitution is called *Gaussian elimination.*

It is named after the famous German mathematician Carl Friedrich Gauss, 1777–1855, who devised it.

Gauss contributed to many branches of mathematics, physics and astronomy. Indeed it was while working on the problem of the orbit of the newly discovered asteroid Ceres that he began to explore numerical analysis and the theory of errors.

EXERCISE 4A

1 For each system of equations:

(i) express it as an augmented matrix

(ii) reduce the matrix, using row operations, to upper triangular form

(iii) using back-substitution, work out the values of the variables.

a $\begin{aligned} x + 2y + z &= 8 \\ 3x + y - 2z &= -1 \\ x + 5y - z &= 8 \end{aligned}$

b $\begin{aligned} 2x + 3y - z &= -1 \\ x - 3y - 2z &= 4 \\ 5x + y + 3z &= 4 \end{aligned}$

c $\begin{aligned} 3x + y \phantom{{}- 3z} &= 5 \\ x + 2y - 3z &= -12 \\ x \phantom{{}+ 2y} + 2z &= 10 \end{aligned}$

d $\begin{aligned} 3x - 4y + z &= 24 \\ x - 2y - 2z &= 7 \\ x + y + z &= 4 \end{aligned}$

e $\begin{aligned} 4x + 2y + z &= 3 \\ x + 3y + 5z &= 3 \\ 2x \phantom{{}+ 3y} + 3z &= 5 \end{aligned}$

f $\begin{aligned} x + y + 5z &= 0 \\ 4x + y - 6z &= -17 \\ x - y - z &= 0 \end{aligned}$

2 A parabola passes through the points (1, 2), (2, 7) and (3, 14). It has an equation of the form $y = ax^2 + bx + c$.

a Use the information to form a 3×3 system of equations.

b Solve the system by Gaussian elimination.

c Write down the equation of the parabola.

3 Archaeologists working on an archaeological dig discover the remains of a circular Roman amphitheatre. Using a suitable set of axes and convenient units, the archaeologists positively identify three points on its circumference: (–2, –1), (–1, 2) and (6, 3).

a Assuming the perimeter has an equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$, form a system of equations in g, f and c.

b Use Gaussian elimination to solve the system and identify the equation of the perimeter.

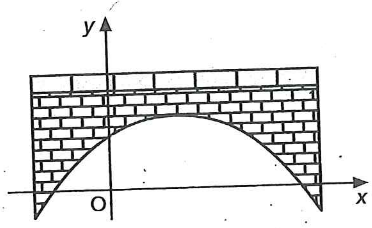
c What is the radius of the amphitheatre?

d $4x + 3y - 2z = 16$
$x - 2y - 3z = -9$
$3x - 5y - 2z = -4$

e $x - y - 3z = 1$
$2x + y - 2z = 9$
$x - 2y + 2z = 5$

f $y + 3z = 3$
$2x + 3z = 10$
$3x + 2y = 0$

2



By working with convenient units, the supporting arch of a bridge can be modelled by the equation $y = a + bx - cx^2$. Three points on this arch have been accurately measured as (1, −3), (2, −28) and (−1, −13).

a Use the data to form a system of equations.

b Solve the system by reducing the augmented matrix to the form $\left(\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{array}\right)$.

c Write down the equation of the arch.

Redundancy and inconsistency in a 3 × 3 system

Consider the system of equations
$$\begin{aligned} x + 2y + 2z &= 11 \\ x - y + 3z &= 8 \\ 4x - y + 11z &= 35 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{array}\right)$$

R2 → R2 − R1
R3 → R3 − 4R1
$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & -9 & 3 & -9 \end{array}\right)$$

R3 → R3 − 3R2
$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The row of zeros tells us that this equation is redundant, so that there is not a unique solution to the system. There is in fact an infinite number of solutions:

given any z then, from row 2: $y = \frac{z-3}{3}$;

from row 1: $x = 11 - 2z - 2y$

which simplifies to $x = \frac{39 - 8z}{3}$

The general solution can be quoted as: $x = \frac{39 - 8z}{3}$; $y = \frac{z-3}{3}$; $z = z$

A particular solution can be found by assigning a value to z.
For example, if $z = 3$ the solution would be $x = 5$, $y = 0$, $z = 3$.

Consider the system of equations

$$\begin{aligned} x + 2y + 2z &= 11 \\ 2x - y + z &= 8 \\ 3x + y + 3z &= 18 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 2 & -1 & 1 & 8 \\ 3 & 1 & 3 & 18 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & -5 & -3 & -15 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Row 3 suggests that $0 = 1$, which tells us that the system of equations is in fact inconsistent and that there are no solutions.

EXERCISE 5

1 Attempt to reduce each of the following systems of equations to upper triangular form.

- Where this is possible quote the unique solution.
- Where there is a redundant equation, find a general solution.
- Where there is inconsistency, declare that there are no solutions.

a $\begin{aligned} 3x + 2y + 5z &= 0 \\ 2x + y - 2z &= 5 \\ 7x + 4y + z &= 10 \end{aligned}$

b $\begin{aligned} x + y - z &= 4 \\ 2x - y + 2z &= -2 \\ x - 3y - 4z &= -1 \end{aligned}$

c $\begin{aligned} 2x - y + 3z &= 6 \\ x + y + 2z &= 7 \\ 4x + y + 7z &= 9 \end{aligned}$

d $\begin{aligned} 2x - 3y + z &= 2 \\ x + y - 3z &= 7 \\ 5x - 2y - z &= 14 \end{aligned}$

e $\begin{aligned} x + 2y - z &= 3 \\ x + 3z &= 5 \\ 4x + 2y + 8z &= 10 \end{aligned}$

f $\begin{aligned} 5x - 3y - z &= -12 \\ 2x + y + 3z &= 3 \\ 20x - y + 13z &= -9 \end{aligned}$

2 The system of equations

$$\begin{aligned} x + 2y - z &= 8 \\ 3x + y + 2z &= -1 \\ x + y + kz &= -6 \end{aligned}$$

has no solutions. What is the value of k?

3 Find the value of k that makes the system of equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y - 2z &= -1 \\ x - y + kz &= 7 \end{aligned}$$

have infinitely many solutions.

4 For what values of d and e will the three equations

$$\begin{aligned} x + 3y - 2z &= 8 \\ 2x + y - 3z &= 5 \\ 7x - 4y + dz &= e \end{aligned}$$

have **a** no solution
b infinitely many solutions
c a unique solution?

4 a (i) $\left(\begin{array}{cc|c} 3 & -2 & 1 \\ 6 & -4 & 3 \end{array}\right)$ (ii) $\begin{pmatrix} 3 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) $0 = a \ (a \neq 0) \Rightarrow$ inconsistency

b (i) $\left(\begin{array}{cc|c} 4 & 2 & 5 \\ 2 & 1 & 3 \end{array}\right)$ (ii) $\left(\begin{array}{cc|c} 4 & 2 & 5 \\ 0 & 0 & 1 \end{array}\right)$

(iii) as **a**

5 a $9 = a - b$; $12 = 4a - 16b$

b $\left(\begin{array}{cc|c} 1 & -1 & 9 \\ 4 & -16 & 12 \end{array}\right)$ $a = 11$; $b = 2$

c $1\frac{1}{2}$ units

6 a $\left(\begin{array}{cc|c} 3 & 1 & 10 \\ 5 & -2 & -9 \end{array}\right)$; (1, 7)

b (i) redundant (ii) inconsistent

c (i) redundancy $\Rightarrow$ same line

(ii) inconsistency $\Rightarrow$ parallel lines.

Exercise 4A (page 127)

1 a (i) $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 3 & 1 & -2 & -1 \\ 1 & 5 & -1 & 8 \end{array}\right)$ (ii) $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -5 & -5 & -25 \\ 0 & 0 & -5 & -15 \end{array}\right)$

(iii) $x = 1$
$y = 2$
$z = 3$

b (i) $\left(\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ 1 & -3 & -2 & 4 \\ 5 & 1 & 3 & 4 \end{array}\right)$ (ii) $\left(\begin{array}{ccc|c} 1 & -3 & -2 & 4 \\ 0 & 9 & 3 & -9 \\ 0 & 0 & -\frac{7}{3} & 0 \end{array}\right)$

(iii) $x = 1$
$y = -1$
$z = 0$

c (i) $\left(\begin{array}{ccc|c} 3 & 1 & 0 & 5 \\ 1 & 2 & -3 & -12 \\ 1 & 0 & 2 & 10 \end{array}\right)$ (ii) $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 10 \\ 0 & 1 & -6 & -25 \\ 0 & 0 & 7 & 28 \end{array}\right)$

(iii) $x = 2$
$y = -1$
$z = 4$

d (i) $\left(\begin{array}{ccc|c} 3 & -4 & 1 & 24 \\ 1 & -2 & -2 & 7 \\ 1 & 1 & 1 & 4 \end{array}\right)$ (ii) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & -5 & 5 \end{array}\right)$

(iii) $x = 5$
$y = -2$
$z = 1$

e (i) $\left(\begin{array}{ccc|c} 4 & 2 & 1 & 3 \\ 1 & 3 & 5 & 3 \\ 2 & 0 & 3 & 5 \end{array}\right)$ (ii) $\left(\begin{array}{ccc|c} 1 & 3 & 5 & 3 \\ 0 & -10 & -19 & -9 \\ 0 & 0 & 1 & 1 \end{array}\right)$

(iii) $x = 1$
$y = -1$
$z = 1$

f (i) $\left(\begin{array}{ccc|c} 1 & 1 & 5 & 0 \\ 4 & 1 & -6 & -17 \\ 1 & -1 & -1 & 0 \end{array}\right)$ (ii) $\left(\begin{array}{ccc|c} 1 & 1 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 17 & 17 \end{array}\right)$

(iii) $x = -2$
$y = -3$
$z = 1$

2 a $a + b + c = 2$, $4a + 2b + c = 7$,
$9a + 3b + c = 14$

b $a = 1$, $b = 2$, $c = -1$

c $y = x^2 + 2x - 1$

3 a $-4g - 2f + c = -5$, $-2g + 4f + c = -5$,
$12g + 6f + c = -45$

b $g = -3$; $f = 1$, $c = -15$;
$x^2 + y^2 - 6x + 2y - 15 = 0$

c $r = 5$

4 a $s + c + g = 185$, $3s + 4c + 2g = 460$,
$2s + 3c + 2g = 375$
$s = 80$, $c = 5$, $g = 100$

b After 1 hr there is 5 seconds of GO phase left.

Exercise 4B (page 128)

1 a $x = 1$, $y = 1$, $z = 1$ **b** $x = 1$, $y = -2$, $z = 1$

c $x = -1$, $y = 3$, $z = 0$ **d** $x = 4$, $y = 2$, $z = 3$

e $x = 5$, $y = 1$, $z = 1$ **f** $x = 2$, $y = -3$, $z = 2$

2 a $a + b - c = -3$, $a + 2b - 4c = -28$,
$a - b - c = -13$

b $a = 2$, $b = 5$, $c = 10$

c $y = 2 + 5x - 100x^2$

Exercise 5 (page 130)

1 a redundant; $x = 9z + 10$, $y = -16z - 15$, $z = z$

b $x = 1$, $y = 2$, $z = -1$

c inconsistent – no solutions

d $x = 3$, $y = 1$, $z = -1$

e inconsistent – no solutions

f x, $\dfrac{17x + 33}{8}$, $\dfrac{-11x - 3}{8}$

2 $k = 0$

3 $k = 9$

4 a $d = -9$ and $e \neq 1$

b $d = -9$ and $e = 1$ **c** $d \neq -9$

Exercise 6 (page 132)

1 a 1, 1, 2, 1 **b** 1, 2, 0, −1

c 1, 5, 6, 2 **d** 1, 3, 5, 1

2 a $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 0$; $S_n = \frac{1}{2}n^2 + \frac{1}{2}n$

b $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$, $d = 0$;
$S_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

c $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = \frac{1}{4}$, $d = 0$;
$S_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$