



# Higher Mathematics

## Trigonometry

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# Trigonometry

## 1 Solving Trigonometric Equations

You should already be familiar with solving some trigonometric equations.

### EXAMPLES

1. Solve  $\sin x^\circ = \frac{1}{2}$  for  $0 < x < 360$ .

$$\sin x^\circ = \frac{1}{2}$$

$$\begin{array}{c} 180^\circ - x^\circ \\ \checkmark S \mid A \checkmark \\ 180^\circ + x^\circ \\ \checkmark T \mid C \checkmark \\ 360^\circ - x^\circ \end{array}$$

Since  $\sin x^\circ$  is positive

First quadrant solution:

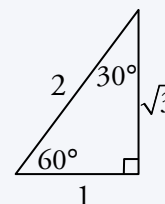
$$\begin{aligned} x &= \sin^{-1}\left(\frac{1}{2}\right) \\ &= 30. \end{aligned}$$

$$x = 30 \quad \text{or} \quad 180 - 30$$

$$x = 30 \quad \text{or} \quad 150.$$

### Remember

The exact value triangle:



2. Solve  $\cos x^\circ = -\frac{1}{\sqrt{5}}$  for  $0 < x < 360$ .

$$\cos x^\circ = -\frac{1}{\sqrt{5}}$$

$$\begin{array}{c} 180^\circ - x^\circ \\ \checkmark S \mid A \checkmark \\ 180^\circ + x^\circ \\ \checkmark T \mid C \checkmark \\ 360^\circ - x^\circ \end{array}$$

Since  $\cos x^\circ$  is negative

$$\begin{aligned} x &= \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \\ &= 63.435 \quad (\text{to 3 d.p.}). \end{aligned}$$

$$x = 180 - 63.435 \quad \text{or} \quad 180 + 63.435$$

$$x = 116.565 \quad \text{or} \quad 243.435.$$

3. Solve  $\sin x^\circ = 3$  for  $0 < x < 360$ .

There are no solutions since  $-1 \leq \sin x^\circ \leq 1$ .

Note that  $-1 \leq \cos x^\circ \leq 1$ , so  $\cos x^\circ = 3$  also has no solutions.



4. Solve  $\tan x^\circ = -5$  for  $0 < x < 360$ .

$$\tan x^\circ = -5$$

$$\begin{array}{c} 180^\circ - x^\circ \\ \checkmark \text{ S } | \text{ A } \checkmark \\ \hline 180^\circ + x^\circ \\ \text{ T } | \text{ C } \checkmark \end{array} \begin{array}{l} x^\circ \\ \\ 360^\circ - x^\circ \end{array}$$

Since  $\tan x^\circ$  is negative

$$\begin{aligned} x &= \tan^{-1}(5) \\ &= 78.690 \text{ (to 3 d.p.)} \end{aligned}$$

$$x = 180 - 78.690 \quad \text{or} \quad 360 - 78.690$$

$$x = 101.310 \quad \text{or} \quad 281.310.$$

### Note

All trigonometric equations we will meet can be reduced to problems like those above. The only differences are:

- the solutions could be required in radians – in this case, the question will not have a degree symbol, e.g. “Solve  $3 \tan x = 1$ ” rather than “ $3 \tan x^\circ = 1$ ”;
- exact value solutions could be required in the non-calculator paper – you will be expected to know the exact values for 0, 30, 45, 60 and 90 degrees.

Questions can be worked through in degrees or radians, but make sure the final answer is given in the units asked for in the question.

### EXAMPLES

5. Solve  $2 \sin 2x^\circ - 1 = 0$  where  $0 \leq x \leq 360$ .

$$2 \sin 2x^\circ = 1$$

$$\sin 2x^\circ = \frac{1}{2}$$

$$\begin{array}{c} 180^\circ - 2x^\circ \\ \checkmark \text{ S } | \text{ A } \checkmark \\ \hline 180^\circ + 2x^\circ \\ \text{ T } | \text{ C } \checkmark \end{array} \begin{array}{l} 2x^\circ \\ \\ 360^\circ - 2x^\circ \end{array}$$

$$0 \leq x \leq 360$$

$$0 \leq 2x \leq 720$$

$$\begin{aligned} 2x &= \sin^{-1}\left(\frac{1}{2}\right) \\ &= 30. \end{aligned}$$

$$2x = 30 \quad \text{or} \quad 180 - 30$$

$$\text{or} \quad 360 + 30 \quad \text{or} \quad 360 + 180 - 30$$

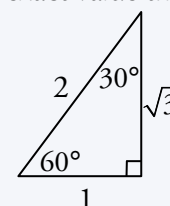
$$\text{or} \quad \cancel{360 + 360 + 30}$$

$$2x = 30 \quad \text{or} \quad 150 \quad \text{or} \quad 390 \quad \text{or} \quad 510$$

$$x = 15 \quad \text{or} \quad 75 \quad \text{or} \quad 195 \quad \text{or} \quad 255.$$

### Remember

The exact value triangle:



### Note

There are more solutions every  $360^\circ$ , since  $\sin(30^\circ) = \sin(30^\circ + 360^\circ) = \dots$ . So keep adding 360 until  $2x > 720$ .

6. Solve  $\sqrt{2} \cos 2x = 1$  where  $0 \leq x \leq \pi$ .

$$\cos 2x = \frac{1}{\sqrt{2}}$$

$$\begin{array}{c} \pi - 2x \\ \hline \text{S} \mid \text{A} \checkmark \\ \hline \text{T} \mid \text{C} \checkmark \\ \hline \pi + 2x \end{array} \quad 0 \leq x \leq \pi$$

$$\begin{array}{c} \pi - 2x \\ \hline \text{S} \mid \text{A} \checkmark \\ \hline \text{T} \mid \text{C} \checkmark \\ \hline \pi + 2x \end{array} \quad 0 \leq 2x \leq 2\pi$$

$$\begin{aligned} 2x &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{\pi}{4}. \end{aligned}$$

$$2x = \frac{\pi}{4} \quad \text{or} \quad 2\pi - \frac{\pi}{4}$$

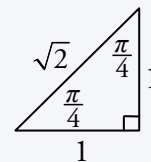
$$\text{or} \quad \cancel{2\pi + \frac{\pi}{4}}$$

$$2x = \frac{\pi}{4} \quad \text{or} \quad \frac{7\pi}{4}$$

$$x = \frac{\pi}{8} \quad \text{or} \quad \frac{7\pi}{8}.$$

**Remember**

The exact value triangle:



7. Solve  $4 \cos^2 x = 3$  where  $0 < x < 2\pi$ .

$$(\cos x)^2 = \frac{3}{4}$$

$$\cos x = \pm \sqrt{\frac{3}{4}}$$

$$\cos x = \pm \frac{\sqrt{3}}{2}$$

$$\begin{array}{c} \checkmark \text{S} \mid \text{A} \checkmark \\ \checkmark \text{T} \mid \text{C} \checkmark \end{array} \quad \text{Since } \cos x \text{ can be positive or negative}$$

$$\begin{aligned} x &= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\pi}{6}. \end{aligned}$$

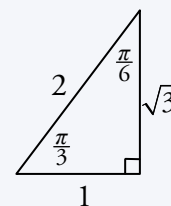
$$x = \frac{\pi}{6} \quad \text{or} \quad \pi - \frac{\pi}{6} \quad \text{or} \quad \pi + \frac{\pi}{6} \quad \text{or} \quad 2\pi - \frac{\pi}{6}$$

$$\text{or} \quad \cancel{2\pi + \frac{\pi}{6}}$$

$$x = \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6} \quad \text{or} \quad \frac{7\pi}{6} \quad \text{or} \quad \frac{11\pi}{6}.$$

**Remember**

The exact value triangle:



8. Solve  $3 \tan(3x^\circ - 20^\circ) = 5$  where  $0 \leq x \leq 360$ .

$$3 \tan(3x^\circ - 20^\circ) = 5$$

$$\tan(3x^\circ - 20^\circ) = \frac{5}{3}$$

$$\begin{array}{c} \text{S} \mid \text{A} \checkmark \\ \hline \checkmark \text{T} \mid \text{C} \end{array}$$

$$0 \leq x \leq 360$$

$$0 \leq 3x \leq 1080$$

$$-20 \leq 3x - 20 \leq 1060$$

$$3x - 20 = \tan^{-1}\left(\frac{5}{3}\right)$$

$$= 59.036 \text{ (to 3 d.p.)}$$

$$3x - 20 = 59.036 \quad \text{or} \quad 180 + 59.036$$

$$\text{or} \quad 360 + 59.036 \quad \text{or} \quad 360 + 180 + 59.036$$

$$\text{or} \quad 360 + 360 + 59.036 \quad \text{or} \quad 360 + 360 + 180 + 59.036$$

$$\text{or} \quad \cancel{360 + 360 + 360 + 59.036}.$$



$$\begin{aligned}
 3x - 20 &= 59.036 \text{ or } 239.036 \text{ or } 419.036 \\
 &\text{or } 599.036 \text{ or } 779.036 \text{ or } 959.036 \\
 3x &= 79.036 \text{ or } 259.036 \text{ or } 439.036 \\
 &\text{or } 619.036 \text{ or } 799.036 \text{ or } 979.036 \\
 x &= 26.35 \text{ or } 86.35 \text{ or } 146.35 \text{ or } 206.35 \text{ or } 266.35 \text{ or } 326.35.
 \end{aligned}$$



9. Solve  $\cos\left(2x + \frac{\pi}{3}\right) = 0.812$  for  $0 < x < 2\pi$ .

$$\begin{aligned}
 \cos\left(2x + \frac{\pi}{3}\right) &= 0.812 & \begin{array}{l|l} \text{S} & \text{A} \checkmark \\ \text{T} & \text{C} \checkmark \end{array} & \begin{array}{l} 0 < x < 2\pi \\ 0 < 2x < 4\pi \\ \frac{\pi}{3} < 2x + \frac{\pi}{3} < 4\pi + \frac{\pi}{3} \\ 1.047 < 2x + \frac{\pi}{3} < 13.614 \text{ (to 3 d.p.)} \end{array} \\
 & & & 2x + \frac{\pi}{3} = \cos^{-1}(0.812) \\
 & & & = 0.623 \text{ (to 3 d.p.)}
 \end{aligned}$$

### Remember

Make sure your calculator uses radians

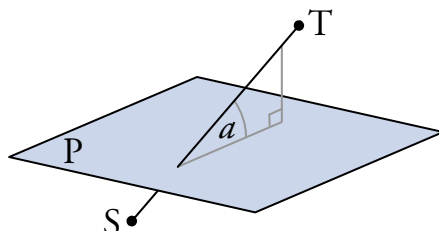
$$\begin{aligned}
 2x + \frac{\pi}{3} &= \cancel{0.623} \text{ or } 2\pi - 0.623 \\
 &\text{or } 2\pi + 0.623 \text{ or } 2\pi + 2\pi - 0.623 \\
 &\text{or } 2\pi + 2\pi + 0.623 \text{ or } \cancel{2\pi + 2\pi + 2\pi - 0.623} \\
 2x + \frac{\pi}{3} &= 5.660 \text{ or } 6.906 \text{ or } 11.943 \text{ or } 13.189 \\
 2x &= 4.613 \text{ or } 5.859 \text{ or } 10.896 \text{ or } 12.142 \\
 x &= 2.307 \text{ or } 2.930 \text{ or } 5.448 \text{ or } 6.071.
 \end{aligned}$$

## 2 Trigonometry in Three Dimensions

It is possible to solve trigonometric problems in three dimensions using techniques we already know from two dimensions. The use of sketches is often helpful.

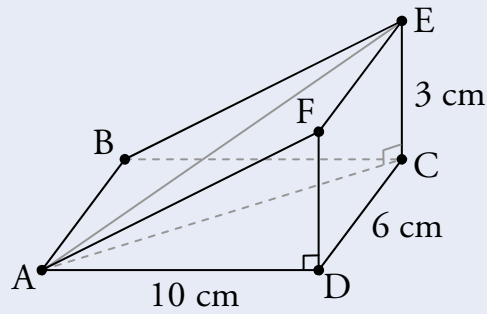
### The angle between a line and a plane

The angle  $a$  between the plane P and the line ST is calculated by adding a line perpendicular to the plane and then using basic trigonometry.



## EXAMPLE

1. The triangular prism ABCDEF is shown below.

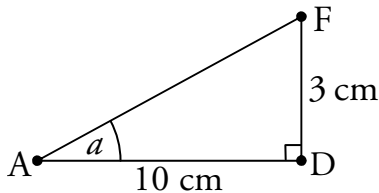


Calculate the acute angle between:

- (a) The line AF and the plane ABCD.  
 (b) AE and ABCD.



(a) Start with a sketch:



$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{10}$$

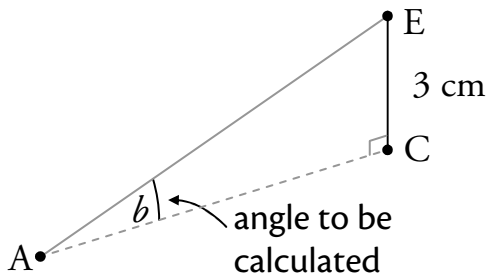
$$a = \tan^{-1}\left(\frac{3}{10}\right)$$

$$= 16.699^\circ \text{ (or } 0.291 \text{ radians) (to 3 d.p.)}$$

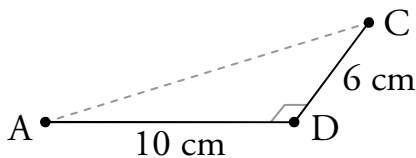
## Note

Since the angle is in a right-angled triangle, it must be acute so there is no need for a CAST diagram.

(b) Again, make a sketch:



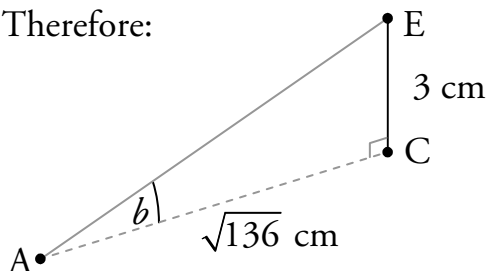
We need to calculate the length of AC first using Pythagoras's Theorem:



$$AC = \sqrt{10^2 + 6^2}$$

$$= \sqrt{136}$$

Therefore:



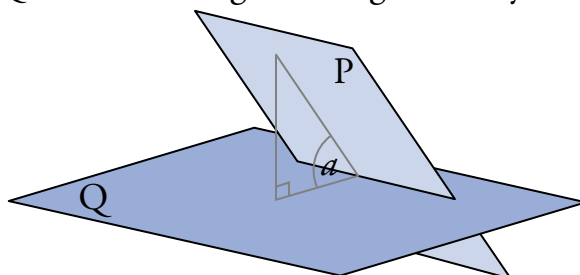
$$\tan b = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{\sqrt{136}}$$

$$b = \tan^{-1}\left(\frac{3}{\sqrt{136}}\right)$$

$$= 14.426^\circ \text{ (or } 0.252 \text{ radians) (to 3 d.p.)}$$

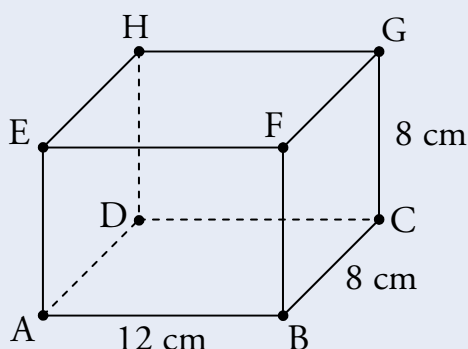
### The angle between two planes

The angle  $a$  between planes P and Q is calculated by adding a line perpendicular to Q and then using basic trigonometry.



#### EXAMPLE

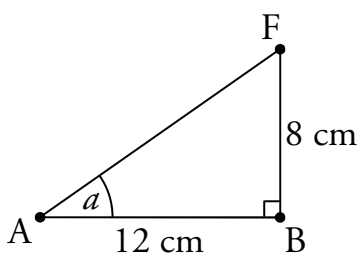
2. ABCDEFGH is a cuboid with dimensions  $12 \times 8 \times 8$  cm as shown below.



- (a) Calculate the size of the angle between the planes AFGD and ABCD.  
 (b) Calculate the size of the acute angle between the diagonal planes AFGD and BCHE.



(c) Start with a sketch:



$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{8}{12}$$

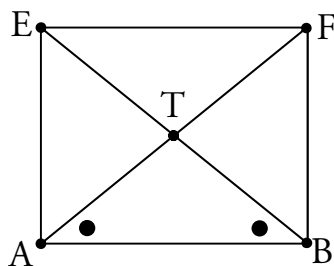
$$a = \tan^{-1}\left(\frac{2}{3}\right)$$

$$= 33.690^\circ \text{ (or } 0.588 \text{ radians) (to 3 d.p.)}$$

#### Note

Angle GDC is the same size as angle FAB.

(d) Again, make a sketch:



Let AF and BE intersect at T

$$\triangle ATB \text{ is isosceles, so } \hat{T}AB = \hat{T}BA = 33.690^\circ$$

$$\begin{aligned} \hat{A}TB &= 180^\circ - (33.690^\circ + 33.690^\circ) \\ &= 112.620^\circ. \end{aligned}$$

So the acute angle is:

$$\begin{aligned} \hat{B}TF = \hat{A}TE &= 180^\circ - 112.620^\circ \\ &= 67.380^\circ \text{ (or } 1.176 \text{ radians) (to 3 d.p.)} \end{aligned}$$

#### Note

The angle could also have been calculated using rectangle DCGH.

### 3 Compound Angles

When we add or subtract angles, the result is a **compound angle**.

For example,  $45^\circ + 30^\circ$  is a compound angle. Using a calculator, we find:

- $\sin(45^\circ + 30^\circ) = \sin(75^\circ) = 0.966$
- $\sin(45^\circ) + \sin(30^\circ) = 1.207$  (both to 3 d.p.).

This shows that  $\sin(A + B)$  is *not* equal to  $\sin A + \sin B$ . Instead, we can use the following identities:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

These are given in the exam in a condensed form:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

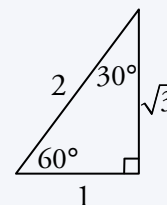
#### EXAMPLES

1. Expand and simplify  $\cos(x^\circ + 60^\circ)$ .

$$\begin{aligned} \cos(x^\circ + 60^\circ) &= \cos x^\circ \cos 60^\circ - \sin x^\circ \sin 60^\circ \\ &= \frac{1}{2} \cos x^\circ - \frac{\sqrt{3}}{2} \sin x^\circ. \end{aligned}$$

#### Remember

The exact value triangle:



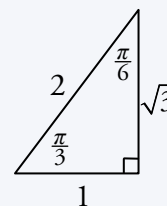
2. Show that  $\sin(a + b) = \sin a \cos b + \cos a \sin b$  for  $a = \frac{\pi}{6}$  and  $b = \frac{\pi}{3}$ .

$$\begin{aligned} \text{LHS} &= \sin(a + b) & \text{RHS} &= \sin a \cos b + \cos a \sin b \\ &= \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) & &= \sin \frac{\pi}{6} \cos \frac{\pi}{3} + \cos \frac{\pi}{6} \sin \frac{\pi}{3} \\ &= \sin\left(\frac{\pi}{2}\right) & &= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}\right) \\ &= 1. & &= \frac{1}{4} + \frac{3}{4} = 1. \end{aligned}$$

Since LHS = RHS, the claim is true for  $a = \frac{\pi}{6}$  and  $b = \frac{\pi}{3}$ .

#### Remember

The exact value triangle:



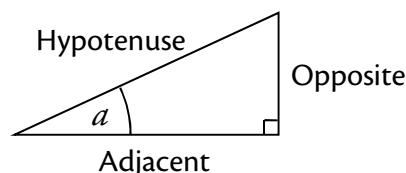


3. Find the exact value of  $\sin 75^\circ$ .

$$\begin{aligned}
 \sin 75^\circ &= \sin(45^\circ + 30^\circ) \\
 &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\
 &= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) \\
 &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\
 &= \frac{\sqrt{3}+1}{2\sqrt{2}} \\
 &= \frac{\sqrt{6}+\sqrt{2}}{4}.
 \end{aligned}$$

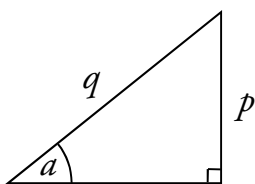
### Finding Trigonometric Ratios

You should already be familiar with the following formulae (SOH CAH TOA).



$$\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos a = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan a = \frac{\text{Opposite}}{\text{Adjacent}}.$$

If we have  $\sin a = \frac{p}{q}$  where  $0 < a < \frac{\pi}{2}$ , then we can form a right-angled triangle to represent this ratio.



Since  $\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{p}{q}$  then:

- the side opposite  $a$  has length  $p$ ;
- the hypotenuse has length  $q$ .

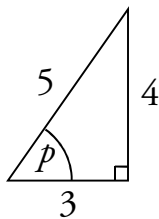
The length of the unknown side can be found using Pythagoras's Theorem.

Once the length of each side is known, we can find  $\cos a$  and  $\tan a$  using SOH CAH TOA.

The method is similar if we know  $\cos a$  and want to find  $\sin a$  or  $\tan a$ .

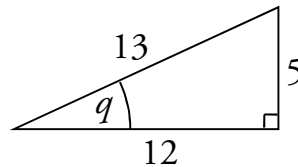
## EXAMPLES

4. Acute angles  $p$  and  $q$  are such that  $\sin p = \frac{4}{5}$  and  $\sin q = \frac{5}{13}$ . Show that  $\sin(p+q) = \frac{63}{65}$ .



$$\sin p = \frac{4}{5}$$

$$\cos p = \frac{3}{5}$$



$$\sin q = \frac{5}{13}$$

$$\cos q = \frac{12}{13}$$

$$\begin{aligned} \sin(p+q) &= \sin p \cos q + \cos p \sin q \\ &= \left(\frac{4}{5} \times \frac{12}{13}\right) + \left(\frac{3}{5} \times \frac{5}{13}\right) \\ &= \frac{48}{65} + \frac{15}{65} \\ &= \frac{63}{65}. \end{aligned}$$

**Note**

Since "Show that" is used in the question, all of this working is required.

## Using compound angle formulae to confirm identities

## EXAMPLES

5. Show that  $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$ .

$$\begin{aligned} &\sin\left(x - \frac{\pi}{2}\right) \\ &= \sin x \cos \frac{\pi}{2} - \cos x \sin \frac{\pi}{2} \\ &= \sin x \times 0 - \cos x \times 1 \\ &= -\cos x. \end{aligned}$$

6. Show that  $\frac{\sin(s+t)}{\cos s \cos t} = \tan s + \tan t$  for  $\cos s \neq 0$  and  $\cos t \neq 0$ .

$$\begin{aligned} \frac{\sin(s+t)}{\cos s \cos t} &= \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t} \\ &= \frac{\sin s \cos t}{\cos s \cos t} + \frac{\cos s \sin t}{\cos s \cos t} \\ &= \frac{\sin s}{\cos s} + \frac{\sin t}{\cos t} \\ &= \tan s + \tan t. \end{aligned}$$

**Remember**

$$\frac{\sin x}{\cos x} = \tan x.$$

## 4 Double-Angle Formulae

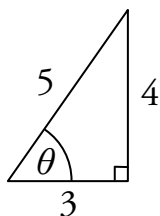
Using the compound angle identities with  $A = B$ , we obtain expressions for  $\sin 2A$  and  $\cos 2A$ . These are called **double-angle formulae**.

$$\begin{aligned}\sin 2A &= 2 \sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \\ &= 1 - 2 \sin^2 A.\end{aligned}$$

Note that these are given in the exam.

### EXAMPLES

1. Given that  $\tan \theta = \frac{4}{3}$ , where  $0 < \theta < \frac{\pi}{2}$ , find the exact value of  $\sin 2\theta$  and  $\cos 2\theta$ .



$$\begin{aligned}\sin \theta &= \frac{4}{5} \\ \cos \theta &= \frac{3}{5}\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ &= 2 \times \frac{4}{5} \times \frac{3}{5} \\ &= \frac{24}{25}.\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25}.\end{aligned}$$

#### Note

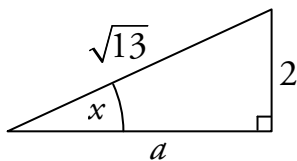
Any of the  $\cos 2A$  formulae could have been used here.

2. Given that  $\cos 2x = \frac{5}{13}$ , where  $0 < x < \pi$ , find the exact values of  $\sin x$  and  $\cos x$ .

Since  $\cos 2x = 1 - 2 \sin^2 x$ ,

$$\begin{aligned}1 - 2 \sin^2 x &= \frac{5}{13} \\ 2 \sin^2 x &= \frac{8}{13} \\ \sin^2 x &= \frac{8}{26} \\ &= \frac{4}{13} \\ \sin x &= \pm \frac{2}{\sqrt{13}}.\end{aligned}$$

We are told that  $0 < x < \pi$ , so only  $\sin x = \frac{2}{\sqrt{13}}$  is possible.



$$a = \sqrt{\sqrt{13}^2 - 2^2} = \sqrt{13 - 4} = \sqrt{9} = 3.$$

$$\text{So } \cos x = \frac{3}{\sqrt{13}}.$$

## 5 Further Trigonometric Equations

We will now consider trigonometric equations where double-angle formulae can be used to find solutions. These equations will involve:

- $\sin 2x$  and either  $\sin x$  or  $\cos x$ ;
- $\cos 2x$  and  $\cos x$ ;
- $\cos 2x$  and  $\sin x$ .

### Remember

The double-angle formulae are given in the exam.

### Solving equations involving $\sin 2x$ and either $\sin x$ or $\cos x$

#### EXAMPLE

1. Solve  $\sin 2x^\circ = -\sin x^\circ$  for  $0 \leq x < 360$ .

$$\begin{aligned}
 2 \sin x^\circ \cos x^\circ &= -\sin x^\circ & \bullet \text{ Replace } \sin 2x \text{ using the double angle formula} \\
 2 \sin x^\circ \cos x^\circ + \sin x^\circ &= 0 & \bullet \text{ Take all terms to one side, making the equation equal to zero} \\
 \sin x^\circ (2 \cos x^\circ + 1) &= 0 & \bullet \text{ Factorise the expression and solve} \\
 \sin x^\circ = 0 & & \\
 x = 0 \text{ or } 180 \text{ or } \cancel{360} & & \\
 2 \cos x^\circ + 1 = 0 & & \\
 \cos x^\circ = -\frac{1}{2} & & \begin{array}{l} \checkmark \text{ S | A} \\ \checkmark \text{ T | C} \end{array} \\
 x = 180 - 60 \text{ or } 180 + 60 & & x = \cos^{-1}\left(\frac{1}{2}\right) \\
 = 120 \text{ or } 240. & & = 60.
 \end{aligned}$$

So  $x = 0$  or  $120$  or  $180$  or  $240$ .

### Solving equations involving $\cos 2x$ and $\cos x$

#### EXAMPLE

2. Solve  $\cos 2x = \cos x$  for  $0 \leq x \leq 2\pi$ .

$$\begin{aligned}
 \cos 2x &= \cos x & \bullet \text{ Replace } \cos 2x \text{ by } 2 \cos^2 x - 1 \\
 2 \cos^2 x - 1 &= \cos x & \bullet \text{ Take all terms to one side, making a quadratic equation in } \cos x \\
 2 \cos^2 x - \cos x - 1 &= 0 & \bullet \text{ Solve the quadratic equation (using factorisation or the quadratic formula)} \\
 (2 \cos x + 1)(\cos x - 1) &= 0 & \\
 2 \cos x + 1 = 0 & & \begin{array}{l} \checkmark \text{ S | A} \\ \checkmark \text{ T | C} \end{array} & \cos x - 1 = 0 \\
 \cos x = -\frac{1}{2} & & & \cos x = 1 \\
 x = \pi - \frac{\pi}{3} \text{ or } \pi + \frac{\pi}{3} & & x = \cos^{-1}\left(\frac{1}{2}\right) & x = 0 \text{ or } 2\pi. \\
 = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} & & = \frac{\pi}{3} & \\
 \text{So } x = 0 \text{ or } \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } 2\pi. & & &
 \end{aligned}$$

Solving equations involving  $\cos 2x$  and  $\sin x$ **EXAMPLE**3. Solve  $\cos 2x = \sin x$  for  $0 < x < 2\pi$ .

$$\cos 2x = \sin x$$

$$1 - 2\sin^2 x = \sin x$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$2\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \quad \text{or} \quad \pi - \frac{\pi}{6}$$

$$= \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}$$

$$\text{So } x = \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6} \quad \text{or} \quad \frac{3\pi}{2}.$$

- Replace  $\cos 2x$  by  $1 - 2\sin^2 x$
- Take all terms to one side, making a quadratic equation in  $\sin x$
- Solve the quadratic equation (using factorisation or the quadratic formula)

$$\sin x + 1 = 0$$

$$\sin x = -1$$

$$x = \frac{3\pi}{2}.$$

$$\begin{array}{c} \checkmark \text{ S } | \text{ A } \checkmark \\ \hline \text{ T } | \text{ C } \end{array}$$

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{6}$$

## 6 Expressing $p\cos x + q\sin x$ in the form $k\cos(x - a)$

An expression of the form  $p\cos x + q\sin x$  can be written in the form  $k\cos(x - a)$  where:

$$k = \sqrt{p^2 + q^2} \quad \text{and} \quad \tan a = \frac{k\sin a}{k\cos a}.$$

The following example shows how to achieve this.

### EXAMPLES



1. Write  $5\cos x^\circ + 12\sin x^\circ$  in the form  $k\cos(x^\circ - a^\circ)$  where  $0 \leq a < 360$ .

#### Step 1

Expand  $k\cos(x - a)$  using the compound angle formula.

$$\begin{aligned} k\cos(x^\circ - a^\circ) \\ &= k\cos x^\circ \cos a^\circ + k\sin x^\circ \sin a^\circ \\ &= k\cos a^\circ \cos x^\circ + k\sin a^\circ \sin x^\circ \end{aligned}$$

#### Step 2

Rearrange to compare with  $p\cos x + q\sin x$ .

$$= \underbrace{k\cos a^\circ}_{5} \cos x^\circ + \underbrace{k\sin a^\circ}_{12} \sin x^\circ$$

#### Step 3

Compare the coefficients of  $\cos x$  and  $\sin x$  with  $p\cos x + q\sin x$ .

$$\begin{aligned} k\cos a^\circ &= 5 \\ k\sin a^\circ &= 12 \end{aligned}$$

#### Step 4

Mark the quadrants on a CAST diagram, according to the signs of  $k\cos a$  and  $k\sin a$ .

$$\begin{array}{c|c} 180^\circ - a^\circ & \text{S} \quad \checkmark \quad \text{A} \quad \checkmark \quad \checkmark \\ \hline 180^\circ + a^\circ & \text{T} \quad \checkmark \quad \text{C} \quad \checkmark \quad 360^\circ - a^\circ \end{array}$$

#### Step 5

Find  $k$  and  $a$  using the formulae above ( $a$  lies in the quadrant marked twice in *Step 4*).

$$\begin{aligned} k &= \sqrt{5^2 + 12^2} & \tan a^\circ &= \frac{k\sin a^\circ}{k\cos a^\circ} \\ &= \sqrt{169} & &= \frac{12}{5} \\ &= 13 & a &= \tan^{-1}\left(\frac{12}{5}\right) \\ & & &= 67.4 \quad (\text{to 1 d.p.}) \end{aligned}$$

#### Step 6

State  $p\cos x + q\sin x$  in the form  $k\cos(x - a)$  using these values.

$$5\cos x^\circ + 12\sin x^\circ = 13\cos(x^\circ - 67.4^\circ)$$



2. Write  $5 \cos x - 3 \sin x$  in the form  $k \cos(x - a)$  where  $0 \leq a < 2\pi$ .

$$\begin{aligned} 5 \cos x - 3 \sin x &= k \cos(x - a) \\ &= k \cos x \cos a + k \sin x \sin a \\ &= (k \cos a) \cos x + (k \sin a) \sin x \end{aligned}$$

$$k \cos a = 5 \qquad k = \sqrt{5^2 + (-3)^2} \qquad \tan a = \frac{k \sin a}{k \cos a} = -\frac{3}{5}$$

$$k \sin a = -3 \qquad = \sqrt{34}$$

First quadrant answer is:

$$\begin{array}{c|c} \pi - a & \text{S} \mid \text{A} \checkmark \\ \hline \checkmark & \text{T} \mid \text{C} \checkmark \checkmark \\ \pi + a & \text{T} \mid \text{C} \checkmark \checkmark \\ & 2\pi - a \end{array}$$

$$\tan^{-1}\left(\frac{3}{5}\right)$$

$$= 0.540 \text{ (to 3 d.p.)}$$

$$\text{So } a = 2\pi - 0.540$$

$$= 5.743 \text{ (to 1 d.p.)}$$

Hence  $a$  is in the fourth quadrant.

$$\text{Hence } 5 \cos x - 3 \sin x = \sqrt{34} \cos(x - 5.743).$$

**Note**

Make sure your calculator is in radian mode.

## 7 Expressing $p \cos x + q \sin x$ in other forms

An expression in the form  $p \cos x + q \sin x$  can also be written in any of the following forms using a similar method:

$$k \cos(x + a), \quad k \sin(x - a), \quad k \sin(x + a).$$

**EXAMPLES**

1. Write  $4 \cos x^\circ + 3 \sin x^\circ$  in the form  $k \sin(x^\circ + a^\circ)$  where  $0 \leq a < 360$ .

$$\begin{aligned} 4 \cos x^\circ + 3 \sin x^\circ &= k \sin(x^\circ + a^\circ) \\ &= k \sin x^\circ \cos a^\circ + k \cos x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \sin x^\circ + (k \sin a^\circ) \cos x^\circ. \end{aligned}$$

$$k \cos a^\circ = 3 \qquad k = \sqrt{4^2 + 3^2} \qquad \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{4}{3}$$

$$k \sin a^\circ = 4 \qquad = \sqrt{25}$$

So:

$$\begin{array}{c|c} 180^\circ - a^\circ & \text{S} \mid \text{A} \checkmark \checkmark \\ \hline \checkmark & \text{T} \mid \text{C} \checkmark \\ 180^\circ + a^\circ & \text{T} \mid \text{C} \checkmark \\ & 360^\circ - a^\circ \end{array}$$

$$\begin{aligned} a &= \tan^{-1}\left(\frac{4}{3}\right) \\ &= 53.1 \text{ (to 1 d.p.)} \end{aligned}$$

Hence  $a$  is in the first quadrant.

$$\text{Hence } 4 \cos x^\circ + 3 \sin x^\circ = 5 \sin(x^\circ + 53.1^\circ).$$



2. Write  $\cos x - \sqrt{3} \sin x$  in the form  $k \cos(x + a)$  where  $0 \leq a < 2\pi$ .

$$\begin{aligned}\cos x - \sqrt{3} \sin x &= k \cos(x + a) \\ &= k \cos x \cos a - k \sin x \sin a \\ &= (k \cos a) \cos x - (k \sin a) \sin x.\end{aligned}$$

$$\begin{aligned}k \cos a &= 1 & k &= \sqrt{1^2 + (-\sqrt{3})^2} & \tan a &= \frac{k \sin a}{k \cos a} = \sqrt{3} \\ k \sin a &= \sqrt{3} & &= \sqrt{1+3} & \text{So:} & \\ \pi - a & \begin{array}{c} \checkmark S | A \checkmark \checkmark \\ \hline \checkmark T | C \checkmark \\ \pi + a \quad \quad \quad 2\pi - a \end{array} & &= \sqrt{4} & a &= \tan^{-1}(\sqrt{3}) \\ & & &= 2 & &= \frac{\pi}{3}.\end{aligned}$$

Hence  $a$  is in the first quadrant.

$$\text{Hence } \cos x - \sqrt{3} \sin x = 2 \cos\left(x + \frac{\pi}{3}\right).$$

## 8 Multiple Angles

We can use the same method with expressions involving the same multiple angle, i.e.  $p \cos(nx) + q \sin(nx)$ , where  $n$  is a constant.

### EXAMPLE



Write  $5 \cos 2x^\circ + 12 \sin 2x^\circ$  in the form  $k \sin(2x^\circ + a^\circ)$  where  $0 \leq a < 360$ .

$$\begin{aligned}5 \cos 2x^\circ + 12 \sin 2x^\circ &= k \sin(2x^\circ + a^\circ) \\ &= k \sin 2x^\circ \cos a^\circ + k \cos 2x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \sin 2x^\circ + (k \sin a^\circ) \cos 2x^\circ.\end{aligned}$$

$$\begin{aligned}k \cos a^\circ &= 12 & k &= \sqrt{12^2 + 5^2} & \tan a^\circ &= \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{5}{12} \\ k \sin a^\circ &= 5 & &= \sqrt{169} & \text{So:} & \\ 180^\circ - a^\circ & \begin{array}{c} \checkmark S | A \checkmark \checkmark \\ \hline \checkmark T | C \checkmark \\ 180^\circ + a^\circ \quad \quad \quad 360^\circ - a^\circ \end{array} & &= 13 & a &= \tan^{-1}\left(\frac{5}{12}\right) \\ & & & & &= 22.6 \text{ (to 1 d.p.).}\end{aligned}$$

Hence  $a$  is in the first quadrant.

$$\text{Hence } 5 \cos 2x^\circ + 12 \sin 2x^\circ = 13 \sin(2x^\circ + 22.6^\circ).$$





## 10 Solving Equations

The method of writing two trigonometric terms as one can be used to help solve equations involving both a  $\sin(nx)$  and a  $\cos(nx)$  term.

### EXAMPLES



1. Solve  $5 \cos x^\circ + \sin x^\circ = 2$  where  $0 \leq x < 360$ .

First, we write  $5 \cos x^\circ + \sin x^\circ$  in the form  $k \cos(x^\circ - a^\circ)$ :

$$\begin{aligned} 5 \cos x^\circ + \sin x^\circ &= k \cos(x^\circ - a^\circ) \\ &= k \cos x^\circ \cos a^\circ + k \sin x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \cos x^\circ + (k \sin a^\circ) \sin x^\circ. \end{aligned}$$

$$k \cos a^\circ = 5 \qquad k = \sqrt{5^2 + 1^2} \qquad \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{1}{5}$$

$$k \sin a^\circ = 1 \qquad = \sqrt{26} \qquad \text{So:}$$

$$\begin{array}{c} 180^\circ - a^\circ \\ \checkmark \text{ S } | \text{ A } \checkmark \checkmark \\ \hline 180^\circ + a^\circ \\ \text{ T } | \text{ C } \checkmark \\ 360^\circ - a^\circ \end{array} \qquad a = \tan^{-1}\left(\frac{1}{5}\right) \\ = 11.3 \text{ (to 1 d.p.)}$$

Hence  $a$  is in the first quadrant.

$$\text{Hence } 5 \cos x^\circ + \sin x^\circ = \sqrt{26} \cos(x^\circ - 11.3^\circ).$$

Now we use this to help solve the equation:

$$\begin{aligned} 5 \cos x^\circ + \sin x^\circ &= 2 \\ \sqrt{26} \cos(x^\circ - 11.3^\circ) &= 2 \\ \cos(x^\circ - 11.3^\circ) &= \frac{2}{\sqrt{26}} \end{aligned} \qquad \begin{array}{c} 180^\circ - x^\circ \\ \text{ S } | \text{ A } \checkmark \\ \hline 180^\circ + x^\circ \\ \text{ T } | \text{ C } \checkmark \\ 360^\circ - x^\circ \end{array}$$

$$\begin{aligned} x - 11.3 &= \cos^{-1}\left(\frac{2}{\sqrt{26}}\right) \\ &= 66.9 \text{ (to 1 d.p.)} \end{aligned}$$

$$x - 11.3 = 66.9 \quad \text{or} \quad 360 - 66.9$$

$$x - 11.3 = 66.9 \quad \text{or} \quad 293.1$$

$$x = 78.2 \quad \text{or} \quad 304.4.$$



2. Solve  $2 \cos 2x + 3 \sin 2x = 1$  where  $0 \leq x < 2\pi$ .

First, we write  $2 \cos 2x + 3 \sin 2x$  in the form  $k \cos(2x - a)$ :

$$\begin{aligned} 2 \cos 2x + 3 \sin 2x &= k \cos(2x - a) \\ &= k \cos 2x \cos a + k \sin 2x \sin a \\ &= (k \cos a) \cos 2x + (k \sin a) \sin 2x. \end{aligned}$$

$$k \cos a = 2 \qquad k = \sqrt{2^2 + (-3)^2} \qquad \tan a = \frac{k \sin a}{k \cos a} = \frac{3}{2}$$

$$k \sin a = 3 \qquad = \sqrt{4 + 9} \qquad \text{So:}$$

$$\begin{array}{c} \pi - a \\ \checkmark S \mid A \checkmark \\ \hline \pi + a \\ T \mid C \checkmark \\ 2\pi - a \end{array} \qquad = \sqrt{13} \qquad a = \tan^{-1}\left(\frac{3}{2}\right) \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad = 0.983 \text{ (to 3 d.p.).}$$

Hence  $a$  is in the first quadrant.

$$\text{Hence } 2 \cos 2x + 3 \sin 2x = \sqrt{13} \cos(2x - 0.983).$$

Now we use this to help solve the equation:

$$\begin{aligned} 2 \cos 2x + 3 \sin 2x &= 1 & \begin{array}{c} \pi - 2x \\ S \mid A \checkmark \\ \hline \pi + 2x \\ T \mid C \checkmark \\ 2\pi - 2x \end{array} & \begin{array}{l} 0 < x < 2\pi \\ 0 < 2x < 4\pi \end{array} \\ \sqrt{13} \cos(2x - 0.983) &= 1 & & \\ \cos(2x - 0.983) &= \frac{1}{\sqrt{13}} & 2x - 0.983 = \cos^{-1}\left(\frac{1}{\sqrt{13}}\right) & \\ & & = 1.290 \text{ (to 3 d.p.).} & \end{aligned}$$

$$2x - 0.983 = 1.290 \text{ or } 2\pi - 1.290$$

$$\text{or } 2\pi + 1.290 \text{ or } 2\pi + 2\pi - 1.290$$

$$\text{or } \underline{2\pi + 2\pi + 1.290}$$

$$2x - 0.983 = 1.290 \text{ or } 4.993 \text{ or } 7.573 \text{ or } 11.276$$

$$2x = 2.273 \text{ or } 5.976 \text{ or } 8.556 \text{ or } 12.259$$

$$x = 1.137 \text{ or } 2.988 \text{ or } 4.278 \text{ or } 6.130.$$





2. Sketch the graph of  $y = \sin x^\circ + \sqrt{3} \cos x^\circ$  for  $0 \leq x \leq 360$ .

First, we write  $\sin x^\circ + \sqrt{3} \cos x^\circ$  in the form  $k \cos(x^\circ - a^\circ)$ :

$$\begin{aligned} \sin x^\circ + \sqrt{3} \cos x^\circ &= k \cos(x^\circ - a^\circ) \\ &= k \cos x^\circ \cos a^\circ + k \sin x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \cos x^\circ + (k \sin a^\circ) \sin x^\circ. \end{aligned}$$

$$k \cos a^\circ = \sqrt{3} \qquad k = \sqrt{1^2 + \sqrt{3}^2} \qquad \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{1}{\sqrt{3}}$$

$$k \sin a^\circ = 1 \qquad = \sqrt{1+3} \qquad \text{So:}$$

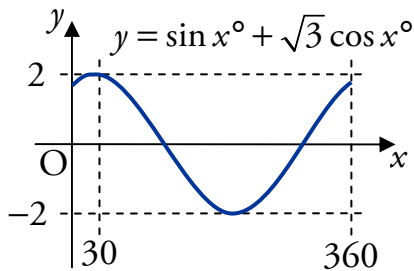
$$180^\circ - a^\circ \quad \begin{array}{c|c} \checkmark S & A \checkmark \checkmark \\ \hline T & C \checkmark \end{array} \quad \begin{array}{c} a^\circ \\ 360^\circ - a^\circ \end{array} \qquad a = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 2 \qquad = 30.$$

Hence  $a$  is in the first quadrant.

Hence  $\sin x^\circ + \sqrt{3} \cos x^\circ = 2 \cos(x^\circ - 30^\circ)$ .

Now we can sketch the graph of  $y = \sin x^\circ + \sqrt{3} \cos x^\circ$ :



3. (a) Write  $5 \sin x^\circ - \sqrt{11} \cos x^\circ$  in the form  $k \sin(x^\circ - a^\circ)$ ,  $0 \leq a < 360$ .



(b) Hence sketch the graph of  $y = 5 \sin x^\circ - \sqrt{11} \cos x^\circ + 2$ ,  $0 \leq x \leq 360$ .

(a) First, we write  $5 \sin x^\circ - \sqrt{11} \cos x^\circ$  in the form  $k \sin(x^\circ - a^\circ)$ :

$$\begin{aligned} 5 \sin x^\circ - \sqrt{11} \cos x^\circ &= k \sin(x^\circ - a^\circ) \\ &= k \sin x^\circ \cos a^\circ - k \cos x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \sin x^\circ - (k \sin a^\circ) \cos x^\circ. \end{aligned}$$

$$\begin{aligned} k \cos a^\circ &= 5 & k &= \sqrt{5^2 + \sqrt{11}^2} & \tan a^\circ &= \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{\sqrt{11}}{5} \\ k \sin a^\circ &= \sqrt{11} & &= \sqrt{25 + 11} & \text{So:} & \\ 180^\circ - a^\circ & \begin{array}{c} \checkmark \\ S \end{array} \begin{array}{c} \checkmark \\ A \end{array} a^\circ & &= \sqrt{36} & a &= \tan^{-1}\left(\frac{\sqrt{11}}{5}\right) \\ 180^\circ + a^\circ & \begin{array}{c} \checkmark \\ T \end{array} \begin{array}{c} \checkmark \\ C \end{array} 360^\circ - a^\circ & &= 6 & &= 33.6 \text{ (to 1 d.p.)} \end{aligned}$$

Hence  $a$  is in the first quadrant.

$$\text{Hence } 5 \sin x^\circ - \sqrt{11} \cos x^\circ = 6 \sin(x^\circ - 33.6^\circ).$$

(b) We can now sketch the graph of

$$y = 5 \sin x^\circ - \sqrt{11} \cos x^\circ + 2 = 6 \sin(x^\circ - 33.6^\circ) + 2:$$

