Advanced Higher Mathematics

Uddingston Grammar

Respecting Learning Caring

Unit 4

Vectors

Outcome 1 – Vectors

Revision of Vectors in 3 dimensions from Higher Level

1. **Position Vector of a point P.**

Relative to an origin O, P is the point (x, y, z) with position vector:-

$$\vec{OP} \text{ or } \underset{\sim}{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{[or } (x, y, z) \text{ is sometimes used]}$$

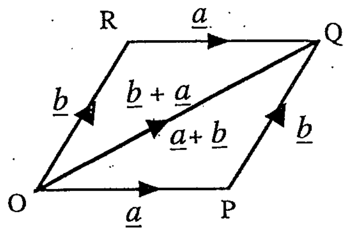
2. **Basic laws**

(a) **The Commutative Law.**

$\underset{\sim}{a}+\underset{\sim}{b}=\underset{\sim}{b}+\underset{\sim}{a}$

Proof :- $\vec{OQ}=\vec{OP}+\vec{PQ} = \underset{\sim}{a}+\underset{\sim}{b}$ and $\vec{OQ}=\vec{OR}+\vec{RQ} = \underset{\sim}{b}+\underset{\sim}{a}$

i.e. $\underset{\sim}{a}+\underset{\sim}{b}=\underset{\sim}{b}+\underset{\sim}{a}$



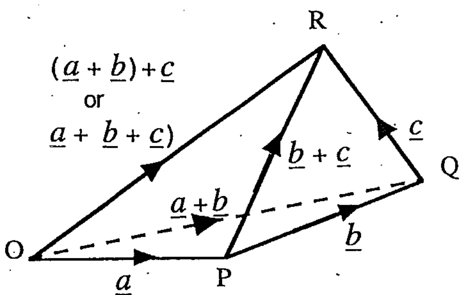
(b) **The Associative Law.**

$(\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}=\underset{\sim}{a}+(\underset{\sim}{b}+\underset{\sim}{c})$

Proof: $\vec{OQ}=\underset{\sim}{a}+\underset{\sim}{b}$ and $\vec{PR}=\underset{\sim}{b}+\underset{\sim}{c}$

$\vec{OR}=\vec{OQ}+\vec{QR} = (\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}$ and $\vec{OR}=\vec{OP}+\vec{PR} = \underset{\sim}{a}+(\underset{\sim}{b}+\underset{\sim}{c})$

i.e. $(\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}=\underset{\sim}{a}+(\underset{\sim}{b}+\underset{\sim}{c})$



(c) The **Zero Vector** or **Identity Vector**

$\underset{\sim}{a}+\underset{\sim}{0}=\underset{\sim}{0}+\underset{\sim}{a}=\underset{\sim}{a}$

(d) The **Negative** of a Vector.

$\underset{\sim}{a}+(-\underset{\sim}{a})=\underset{\sim}{0}$

(e) **Multiplication by a Scalar, *k*.**

If $\underset{\sim}{a}$ is a non-zero vector and k a non-zero number, then

(i) $|k\underset{\sim}{a}|$ is k times $|\underset{\sim}{a}|$.

(ii) if $k > 0$, $k\underset{\sim}{a}$ is parallel to $\underset{\sim}{a}$ and in the same direction.

(iii) if $k < 0$, $k\underset{\sim}{a}$ is parallel to $\underset{\sim}{a}$ and in the opposite direction.

(f) **Unit vector**

A unit vector is one whose magnitude (length) is <u>one unit</u>.

$\underset{\sim}{i}$ is a unit vector in the direction of O*x*.

$\underset{\sim}{j}$ is a unit vector in the direction of O*y*.

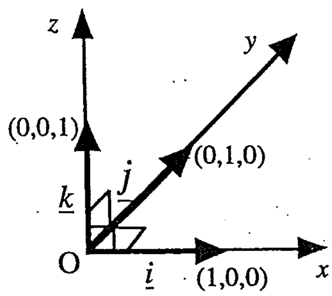
$\underset{\sim}{k}$ is a unit vector in the direction of O*z*.

The position vector of any point can be given in terms of $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$.

e.g. If P is the point (1, 2, 3), then

$$\underset{\sim}{p}=\underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k}$$

or $$\underset{\sim}{p}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$



(g) The length, or **Magnitude**, of a vector.

If $\vec{AB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $|\vec{AB}| = \sqrt{x^2 + y^2 + z^2}$

The distance between A(x_1, y_1, z_1) and B(x_2, y_2, z_2).

$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

$$|\vec{AB}| = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

N.B. The **unit vector** in the direction of a vector $\underline{u}$ is given by the formula $\dfrac{\underline{u}}{|\underline{u}|}$

(h) Subtraction of vectors:-

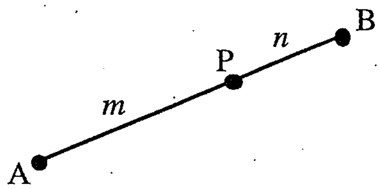
If the position vector of P is $\underline{p}$ and the position vector of Q is $\underline{q}$ then $\vec{PQ} = \underline{q} - \underline{p}$

3. **Formulae:-**

(a) The **Section Formula**:-

If P divides AB in the ratio $m{:}n$,

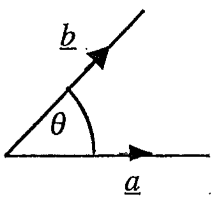
then $\dfrac{\vec{AP}}{\vec{PB}} = \dfrac{m}{n}$ giving $\underline{p} = \dfrac{m\underline{b} + n\underline{a}}{m+n}$



If P is the mid-point of AB, then $\underline{p} = \dfrac{\underline{a} + \underline{b}}{2}$

(b) The **Scalar (Dot) Product**

(i) $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$
where θ is the angle between $\underline{a}$ and $\underline{b}$.
$\underline{a}.\underline{a} = |\underline{a}|^2$ since $\cos\theta = 1$.



(ii) If $\underline{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

then $\underline{a}.\underline{b} = x_1x_2 + y_1y_2 + z_1z_2$

(iii) $\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} = \dfrac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}\sqrt{(x_2^2 + y_2^2 + z_2^2)}}$

(iv) $\underline{a}$ and $\underline{b}$ are perpendicular <=> $\underline{a}.\underline{b} = 0$.
Note also that :-
Parallel vectors $\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$ but
Perpendicular vectors $\underline{i}.\underline{j} = \underline{j}.\underline{k} = \underline{k}.\underline{i} = 0$.

(v) The **Distributive Law**

$\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$

Examples

1. If $\underline{a} = 5\underline{i} + 3\underline{j} + 7\underline{k}$ and $\underline{b} = 2\underline{i} - 8\underline{j} + 4\underline{k}$, find the angle between $\underline{a}$ and $\underline{b}$.

$$\underline{a}.\underline{b} = \begin{pmatrix}5\\3\\7\end{pmatrix}.\begin{pmatrix}2\\-8\\4\end{pmatrix} = 14$$

$|\underline{a}| = \sqrt{(25+9+49)} = \sqrt{83}$ $\quad |\underline{b}| = \sqrt{(4+64+16)} = \sqrt{84}$

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} = \frac{14}{\sqrt{83}\sqrt{84}} \Rightarrow \theta = 80{\cdot}3°$$

2. Find the unit vectors which make an angle of 45° with the vector $\underline{a} = 2\underline{i} + 2\underline{j} - \underline{k}$ and an angle of 60° with the vector $\underline{b} = \underline{j} - \underline{k}$.

Let the unit vector be $\underline{u} = x\underline{i} + y\underline{j} + z\underline{k}$

$\underline{u}.\underline{a} = |\underline{u}||\underline{a}|\cos\theta$

i.e. $2x + 2y - z = 1\text{x}3\text{x}\cos 45° \Rightarrow 2x + 2y - z = \dfrac{3}{\sqrt{2}}$ .....(1)

$\underline{u}.\underline{b} = |\underline{u}||\underline{b}|\cos\theta$

i.e. $y - z = 1\text{x}\sqrt{2}\text{x}\cos 60° \Rightarrow y - z = \dfrac{\sqrt{2}}{2}$ .....(2)

Since $\underline{u}$ is a unit vector then $x^2 + y^2 + z^2 = 1$ ...... (3)

From (1) and (2) $2x + 2y - z = \dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$ ...... (1)

$y - z = \dfrac{\sqrt{2}}{2}$ ...... (2)

Subtract $2x + y = \sqrt{2}$ or $y = \sqrt{2} - 2x$

From (1) and (2) $2x + 2y - z = \dfrac{3\sqrt{2}}{2}$ ...... (1)

$2y - 2z = \sqrt{2}$ ...... (2)

Subtract $2x + z = \dfrac{\sqrt{2}}{2}$ or $z = \dfrac{\sqrt{2}}{2} - 2x$

Substitute in (3)

$$x^2 + \left(\sqrt{2} - 2x\right)^2 + \left(\frac{\sqrt{2}}{2} - 2x\right)^2 = 1 \Rightarrow 9x^2 - 6\sqrt{2}x + \frac{3}{2} = 0$$

$6x^2 - 4\sqrt{2}x + 1 = 0 \Rightarrow x = \dfrac{1}{3\sqrt{2}}$ or $\dfrac{1}{\sqrt{2}}$ (quadratic formula)

$\Rightarrow y = \dfrac{4}{3\sqrt{2}}$ or 0

$\Rightarrow z = \dfrac{1}{3\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}}$

Hence $\underline{u} = \left(\dfrac{1}{3\sqrt{2}}, \dfrac{4}{3\sqrt{2}}, \dfrac{1}{3\sqrt{2}}\right)$ or $\left(\dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}}\right)$

Exercise 1

1. Which of the following expressions represent vectors and which represent scalars?

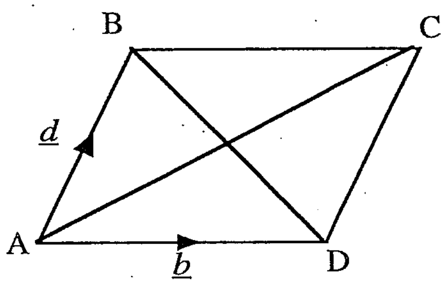
   (a) $\underline{b}.\underline{c} + \underline{c}.\underline{a} + \underline{a}.\underline{b}$ (b) $(\underline{b}.\underline{c})\underline{a} + (\underline{c}.\underline{a})\underline{b} + (\underline{a}.\underline{b})\underline{c}$

   (c) $((\underline{b}.\underline{c})(\underline{c}.\underline{a}))\underline{a}$ (d) $[(\underline{b}.\underline{c})\underline{c} + (\underline{b}.\underline{a})\underline{a}].(\underline{b} + 2\underline{a})$

   Evaluate these when $\underline{a} = \underline{i} + \underline{k}$, $\underline{b} = \underline{i} + \underline{j} + 2\underline{k}$ and $\underline{c} = 2\underline{j} + \underline{k}$.

2. A, B and C have coordinates A(1, 6, –2), B(2, 5, 4), C(4, 6, –3). Find the lengths of the sides, cosines of the angles and the area of triangle ABC.

3. Prove, that in any parallelogram, the sum of the squares on the diagonals is equal to twice the sum of the squares on two adjacent sides.



   i.e. Prove that $|\overrightarrow{BD}|^2 + |\overrightarrow{AC}|^2 = 2|\overrightarrow{AB}|^2 + 2|\overrightarrow{AD}|^2$ by expressing $\overrightarrow{BD}$ as $\underline{b} - \underline{d}$ and $\overrightarrow{AC}$ as $\underline{b} + \underline{d}$

4. Find the two **unit** vectors which make an angle of 45° with the vectors $\underline{a} = \underline{i}$ and $\underline{b} = \underline{k}$.

5. (a) Find the unit vectors which make an angle of 45° with the vector $\underline{a} = -\underline{i} + \underline{k}$ and an angle of 60° with the vector $\underline{b} = -2\underline{i} + 2\underline{j} + \underline{k}$.

   (b) Show that these two unit vectors are perpendicular.

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough) (No reference)

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 62 Exercise 2C Question 1 – 18
Page 412 Exercise 17A Question 1 – 14

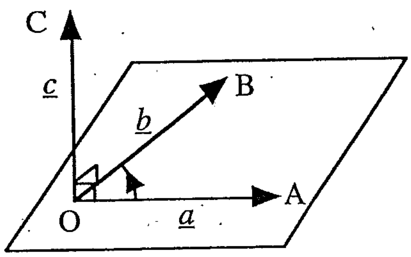
The Vector Product.

Introduction

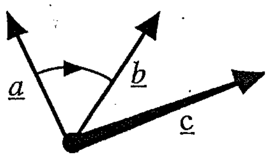
NOT IN COURSE

Right-Hand systems and Left-Hand systems.

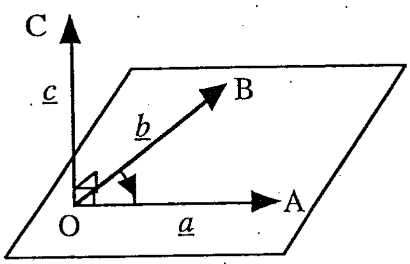
Let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be three, non-zero, non-coplanar vectors.



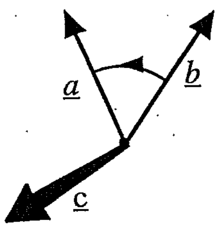
$[\underline{a}, \underline{b}, \underline{c}]$ is said to be a right-hand system if a observer at C sees an anti-clockwise rotation that takes $\vec{OA}$ on to $\vec{OB}$.



Imagine $\underline{c}$ as a screw. Seen from below, a right-hand rotation of the screw takes C into the page.

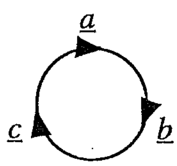


$[\underline{b}, \underline{a}, \underline{c}]$ is said to be a left-hand system if a observer at C sees a clockwise rotation that takes $\vec{OB}$ on to $\vec{OA}$.



Imagine $\underline{c}$ as a screw. Seen from below, a right-hand rotation of the screw takes C out of the page.

Note : The order in which the vectors are written plays a part.
$[\underline{a}, \underline{b}, \underline{c}]$ is a R.H. system.
Cyclic interchange of $\underline{a}$, $\underline{b}$ and $\underline{c}$ is also a R.H. system.



$[\underline{a}, \underline{b}, \underline{c}]$, $[\underline{b}, \underline{c}, \underline{a}]$, $[\underline{c}, \underline{a}, \underline{b}]$ are R.H. systems.
but
$[\underline{a}, \underline{c}, \underline{b}]$, $[\underline{b}, \underline{a}, \underline{c}]$, $[\underline{c}, \underline{b}, \underline{a}]$ are L.H. systems.

The Vector Product.

Definition.

The Vector Product is denoted by $\underline{a} \times \underline{b}$ which reads as "a cross b" and is defined by:-

(i) magnitude (length) $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\theta$ where θ is the angle between $\underline{a}$ and $\underline{b}$.

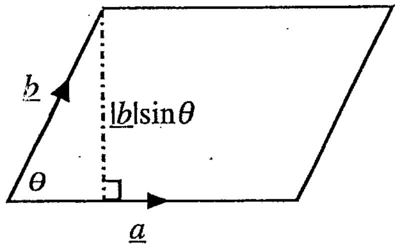
(ii) $\underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$.

(iii) $[\underline{a}, \underline{b}, \underline{a} \times \underline{b}]$ form a R.H. system.

Note (a) $\underline{a} \times \underline{b} = \underline{0} \iff \underline{a}$ is parallel to $\underline{b}$ or either $\underline{a}$ or $\underline{b}$ is zero.

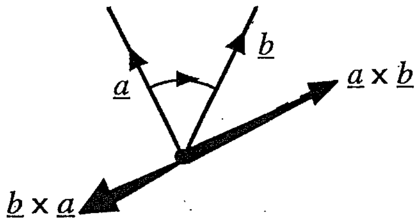
(b) $|\underline{a} \times \underline{b}|$ is the area of a parallelogram with sides determined by $\underline{a}$ and $\underline{b}$.

Area = base x height
$= |\underline{a}||\underline{b}|\sin\theta$.

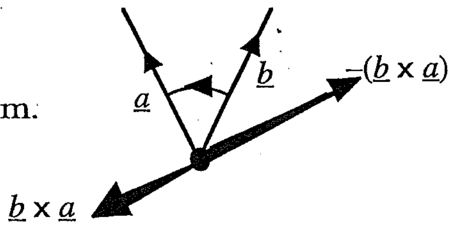


(c) If $\underline{a}$ and $\underline{b}$ are non-zero vectors, the following statements are equivalent :
(i) $\underline{a} \times \underline{b} = \underline{0}$.
(ii) $\underline{a}$ is parallel to $\underline{b}$.
(iii) $\underline{a} = k\underline{b}$. (a linear multiple of $\underline{b}$.)

(d) $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$
Proof :
If $[\underline{a}, \underline{b}, \underline{a} \times \underline{b}]$ is a R.H. system
then $[\underline{b}, \underline{a}, \underline{b} \times \underline{a}]$ is a L.H. system.



So $[\underline{b}, \underline{a}, -(\underline{b} \times \underline{a})]$ is a R.H. system.
i.e. $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$.



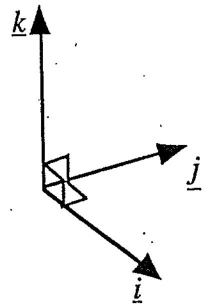
(e) $k\underline{a} \times \underline{b} = k(\underline{a} \times \underline{b})$, $k\underline{a} \times l\underline{b} = kl(\underline{a} \times \underline{b})$

(f) **The distributive law:-**

$\underline{a} \times (\underline{b} + \underline{c}) = (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c})$
$(\underline{b} + \underline{c}) \times \underline{a} = (\underline{b} \times \underline{a}) + (\underline{c} \times \underline{a})$

The Vector Product in component form

Let $\underline{i}$, $\underline{j}$ and $\underline{k}$ be unit vectors, mutually perpendicular to form a R.H. system.



Therefore $\underline{i} \times \underline{j} = \underline{k}$

$\underline{i} \times \underline{k} = -\underline{j}$

$\underline{j} \times \underline{i} = -\underline{k}$

$\underline{j} \times \underline{k} = \underline{i}$

$\underline{k} \times \underline{i} = \underline{j}$

$\underline{k} \times \underline{j} = -\underline{i}$

Also $\underline{i} \times \underline{i} = \underline{0}$

$\underline{j} \times \underline{j} = \underline{0}$

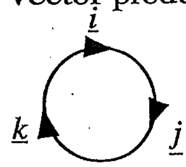
$\underline{k} \times \underline{k} = \underline{0}$

These results can be summarised in a table.

$\times$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	$-\underline{j}$
$\underline{j}$	$-\underline{k}$	$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$	$-\underline{i}$	$\underline{0}$

or

If moving in a clockwise direction, the vector product is positive.
If moving in an anti - clockwise direction, the vector product is negative.



Let $\underline{a} = x_1\underline{i} + y_1\underline{j} + z_1\underline{k}$ and $\underline{b} = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$

then $\underline{a} \times \underline{b} = (x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) \times (x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$

by the distributive law

$= x_1\underline{i}\times(x_2\underline{i} + y_2\underline{j} + z_2\underline{k}) + y_1\underline{j}\times(x_2\underline{i} + y_2\underline{j} + z_2\underline{k}) + z_1\underline{k}\times(x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$

$= x_1\underline{i}\times x_2\underline{i} + x_1\underline{i}\times y_2\underline{j} + x_1\underline{i}\times z_2\underline{k} + y_1\underline{j}\times x_2\underline{i} + y_1\underline{j}\times y_2\underline{j} + y_1\underline{j}\times z_2\underline{k} + z_1\underline{k}\times x_2\underline{i} + z_1\underline{k}\times y_2\underline{j} + z_1\underline{k}\times z_2\underline{k}$

$= x_1x_2\underline{i}\times\underline{i} + x_1y_2\underline{i}\times\underline{j} + x_1z_2\underline{i}\times\underline{k} + x_2y_1\underline{j}\times\underline{i} + y_1y_2\underline{j}\times\underline{j} + y_1z_2\underline{j}\times\underline{k} + x_2z_1\underline{k}\times\underline{i} + y_2z_1\underline{k}\times\underline{j} + z_1z_2\underline{k}\times\underline{k}$

$= x_1x_2(\underline{0}) + x_1y_2(\underline{k}) + x_1z_2(-\underline{j}) + x_2y_1(-\underline{k}) + y_1y_2(\underline{0}) + y_1z_2(\underline{i}) + x_2z_1(\underline{j}) + y_2z_1(-\underline{i}) + z_1z_2(\underline{0})$

$= x_1y_2(\underline{k}) + x_1z_2(-\underline{j}) + x_2y_1(-\underline{k}) + y_1z_2\underline{i} + x_2z_1\underline{j} + y_2z_1(-\underline{i})$

$= (y_1z_2 - y_2z_1)\underline{i} + (x_2z_1 - x_1z_2)\underline{j} + (x_1y_2 - x_2y_1)\underline{k}$

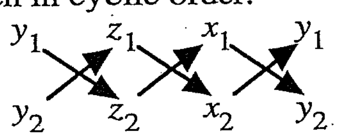
These components can be found more easily by rearranging the components of $\underline{a}$ and $\underline{b}$ under the unit vectors.

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

The component in the direction $\underline{i}$ is found by evaluating the **determinant** formed by the components of $\underline{j}$ and $\underline{k}$. Similarly, the component in the direction $\underline{j}$ is found by evaluating the **determinant** formed by the components of $\underline{i}$ and $\underline{k}$ and the component in the direction $\underline{k}$ is found by evaluating the **determinant** formed by the components of $\underline{i}$ and $\underline{j}$.

Alternatively, write the components of the two vectors one under the other, starting with the second and ending with the second, the components being written in cyclic order.

i.e.



Downward arrows positive.

Upward arrows negative.

Examples 1. If $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$, find (i) $\underline{a} \times \underline{b}$ (ii) $\underline{b} \times \underline{a}$

(i) $$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= (2\text{x}1 - 3\text{x}(-1))\underline{i} - (1\text{x}1 - 3\text{x}2)\underline{j} + (1\text{x}(-1) - 2\text{x}2)\underline{k}$$
$$= 5\underline{i} + 5\underline{j} - 5\underline{k}$$

or

$$\begin{array}{lcccc} \underline{a} & 2 & 3 & 1 & 2 \\ \underline{b} & -1 & 1 & 2 & -1 \end{array}$$

$$\underline{a} \times \underline{b} = (2 - (-3))\underline{i} + (6 - 1)\underline{j} + (-1 - 4)\underline{k}$$
$$= 5\underline{i} + 5\underline{j} - 5\underline{k}$$

(ii) $$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= (-1\text{x}3 - 2\text{x}1)\underline{i} - (2\text{x}3 - 1\text{x}1)\underline{j} + (2\text{x}2 - 1\text{x}(-1))\underline{k}$$
$$= -5\underline{i} - 5\underline{j} + 5\underline{k}$$

2. If $\underline{a} = 3\underline{i} + 5\underline{j} + 7\underline{k}$, $\underline{b} = \underline{i} + \underline{k}$ and $\underline{c} = 2\underline{i} - \underline{j} + 3\underline{k}$,
verify that $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$
(This is known as the **Vector triple product).**

$$\begin{array}{lcccc} \underline{b} & 0 & 1 & 1 & 0 \\ \underline{c} & -1 & 3 & 2 & -1 \end{array} \qquad \underline{b} \times \underline{c} = \underline{i} - \underline{j} - \underline{k}$$

$$\begin{array}{lcccc} \underline{a} & 5 & 7 & 3 & 5 \\ \underline{b} \times \underline{c} & -1 & -1 & 1 & -1 \end{array} \qquad \underline{i} \times (\underline{b} \times \underline{c}) = 2\underline{i} + 10\underline{j} - 8\underline{k}$$

$(\underline{a}.\ \underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$

$$= (6 - 5 + 21)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - (3 + 0 + 7)\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 22\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 10\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -8 \end{pmatrix}$$

i.e. $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$

3. Find the area of the triangle with vertices A(1,3,–2), B(4,3,0) and C(2,1,1).

Area of a triangle $= \frac{1}{2}$(Area of a parallelogram) $= \frac{1}{2}|\vec{CA} \times \vec{CB}|$

$$\vec{CA} = \underline{a} - \underline{c} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix},\ \vec{CB} = \underline{b} - \underline{c} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix},\ \vec{CA} \times \vec{CB} = \begin{pmatrix} 4 \\ -7 \\ -6 \end{pmatrix}$$

Area of a triangle $= \frac{1}{2}\sqrt{(16 + 49 + 36)} = \frac{1}{2}\sqrt{101}$

4. Find a unit vector perpendicular to both $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + 2\underline{k}$:
The vector $\underline{n} = \underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$.
Divide $\underline{n}$ by $|\underline{n}|$ to obtain a unit vector that has the same direction as $\underline{n}$.

Then $\underline{u} = \dfrac{\underline{n}}{|\underline{n}|} = \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$\underline{a} \times \underline{b} = \underline{i} - 5\underline{j} - 3\underline{k}$ from

$\underline{a}$ 1 −1 2 1
$\underline{b}$ −1 2 1 −1

$|\underline{a} \times \underline{b}| = \sqrt{(1^2 + (-5)^2 + (-3)^2)} = \pm\sqrt{35}$

$\underline{u} = \pm\dfrac{1}{\sqrt{35}}(\underline{i} - 5\underline{j} - 3\underline{k})$

Exercise 2

1. If $\underline{a} = 3\underline{i} + 2\underline{j} - \underline{k}$, $\underline{b} = \underline{i} - \underline{j} - 2\underline{k}$ and $\underline{c} = 4\underline{i} - 3\underline{j} + 4\underline{k}$, evaluate
(a) $\underline{a} \times (\underline{b} \times \underline{c})$ (b) $(\underline{a} \times \underline{b}) \times \underline{c}$ (c) $(\underline{a} \times \underline{b}).(\underline{a} \times \underline{c})$
(d) $(\underline{a} \times \underline{b}).(\underline{b} \times \underline{c})$ (e) $[\underline{a} \times (\underline{b} \times \underline{c})].\underline{c}$

2. If $\underline{a} = 3\underline{i} + 2\underline{j} + 5\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} + 2\underline{k}$ and $\underline{c} = 2\underline{i} + \underline{j} + 10\underline{k}$, find
(a) $\underline{a} \times \underline{b}$ (b) $(\underline{a} \times \underline{b}).\underline{c}$ (c) $\underline{b}.(\underline{a} \times \underline{c})$

3. If $\underline{a} = 3\underline{i} + \underline{j} + 2\underline{k}$, $\underline{b} = 2\underline{j} - \underline{k}$ and $\underline{c} = \underline{i} + \underline{j} + \underline{k}$ and $\underline{d} = \underline{b} \times (\underline{c} \times \underline{a}) + (\underline{a}.\underline{c})\underline{a}$, show that $\underline{b}$ is perpendicular to $\underline{d}$.

4. Find a vector perpendicular to each of the vectors $\underline{a} = 4\underline{i} - 2\underline{j} + 3\underline{k}$ and $\underline{b} = 5\underline{i} + \underline{j} - 4\underline{k}$.

5. If $\underline{a} = \underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$, find
(a) $\underline{a} \times \underline{b}$ (b) $\underline{a} \times (\underline{a} + \underline{b})$ and show that (c) $\underline{a}.(\underline{a} \times \underline{b}) = 0$

6. Find the unit vectors perpendicular to both $\underline{a} = 4\underline{i} - \underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$.

7. Find the area of triangle ABC where A(4, −8, −13), B(5, −2, −3) and C(5, 4, 10).

8. Prove algebraically the vector triple product
$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough) - No reference.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning) - No reference

The Equation of a Straight Line.

(i) In Vector Form.

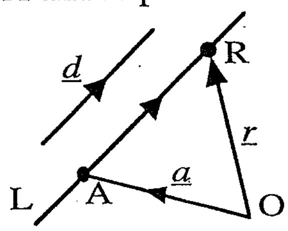
Let $\underline{d} \neq \underline{0}$ be a fixed vector. Let A, with position vector $\underline{a}$, be a fixed point and R, with position vector $\underline{r}$ be a variable point.
Let R lie on a straight line L which passes through A and is parallel to $\underline{d}$.

Since R lies on L, $\overrightarrow{AR}$ is parallel to $\underline{d}$ (or is zero).

$\overrightarrow{AR} = t\underline{d}$ (where t is some parameter).

$\underline{r} - \underline{a} = t\underline{d}$

i.e. $\underline{r} = \underline{a} + t\underline{d}$



This is the **vector equation** of the line L through A parallel to $\underline{d}$.
The scalar t is a parameter and may take any real value including zero.
The vector $\underline{d}$ is called a direction vector of the line.
The cosines of the angles between the vector $\underline{d}$ and the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ are called the **Direction Cosines** of $\underline{d}$.

(ii) In Parametric Form.

If $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{d} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ then the vector equation becomes

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\begin{pmatrix} l \\ m \\ n \end{pmatrix}$, equating components gives

$$\begin{aligned} x &= a + tl \\ y &= b + tm \\ z &= c + tn \end{aligned}$$

These are the **Parametric Equations** of the line.

(iii) In Symmetrical or Cartesian Form.

We can eliminate the parameter, t, to obtain the following:-

From $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$ $(= t)$ $\Rightarrow$ $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$ is called the

Symmetrical Form of the equ^{n} of a line through (a,b,c) in the dir^{n} $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

Note : If any of the direction vectors is **zero**, the parametric equations should be used.

Finding the equation of a line given 2 points on the line.

A particular straight line in space can be precisely specified in a variety of ways :

(a) by means of two points,

(b) by means of one point on the line and the direction vector of the line,

(c) as the intersection of two planes. (See later)

Examples

1. Find the equation of the line joining the points A(1, 0, 2) and B(2, 1, 0)

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$ i.e. the direction vector $\begin{pmatrix}l\\m\\n\end{pmatrix}$ of the line.

A(1, 0, 2) lies on the line.

Substitute in $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$

$\frac{x-1}{1} = \frac{y-0}{1} = \frac{z-2}{-2}$ in symmetrical form ....(1)

$x = 1 + t,\ y = t,\ z = 2 - 2t$ in parametric form.

Note B(2, 1, 0) also lies on the line, giving

$\frac{x-2}{1} = \frac{y-1}{1} = \frac{z}{-2}$ in symmetrical form ....(2)

$x = 2 + t, y = 1 + t, z = -2t$ in parametric form.

This illustrates that the equation of a line is **not** unique. However, equation (1) can be reduced to equation (2) by subtracting 1 from each part.

$\frac{x-1}{1} - 1 = \frac{y-0}{1} - 1 = \frac{z-2}{-2} - 1 \quad => \quad \frac{x-2}{1} = \frac{y-1}{1} = \frac{z}{-2}$

An alternative check, to show that the lines are the same, is

(i) a point on one line lies on the other and

(ii) their direction vectors are parallel.

e.g. A(1, 0, 2) satisfies both equations and Direction Vectors are the same in both equations.

2. Find the symmetrical form of the equation of the line through the point (6, 3, –5).

(a) in direction $\begin{pmatrix}4\\-8\\7\end{pmatrix}$ (b) parallel to the line $\frac{x}{3} = \frac{y-10}{-2} = \frac{z+8}{13}$

=> (a) by inspection $\frac{x-6}{4} = \frac{y-3}{-8} = \frac{z+5}{7}$

=> (b) both lines must have the same direction vector $\begin{pmatrix}3\\-2\\13\end{pmatrix}$

i.e. $\frac{x-6}{3} = \frac{y-3}{-2} = \frac{z+5}{13}$

Exercise 3

1. Find the symmetrical and vector equation of the line through the point (5, –2, 6) in the direction $\begin{pmatrix}3\\1\\4\end{pmatrix}$.

2. Find the symmetrical and vector equation of the line through the point (2, –3, –7) and parallel to the line $\frac{x+5}{7}=\frac{y-13}{3}=\frac{z-4}{-2}$.

3. Find the symmetrical equations of the lines joining the following pairs of points:
   (a) (3, 2, –7), (5, –13, –4) (b) (–8, –13, –9), (12, 7, 1)
   (c) (3, 0, 0), (0, 0, 5) (d) (0, 0, 0), (–10, 4, –6)

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough) (no reference)

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 420 Exercise 17C Questions 1 – 11

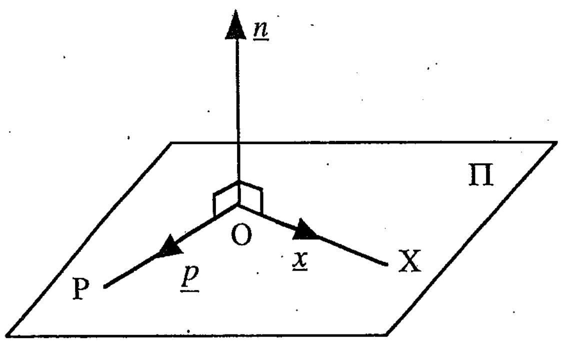
The Equation of a Plane

The Normal Vector.

The normal vector, $\underline{n}$, of a plane is perpendicular to the plane if it is perpendicular to every vector which lies in the plane, Π.

The equation of a plane:-

(i) in the Scalar Product Form of the Vector Equation.



Let $\underline{n} \neq \underline{0}$ be a fixed vector and let P with position vector $\underline{p}$ be a fixed point.

Let Π denote the plane through P normal to $\underline{n}$.

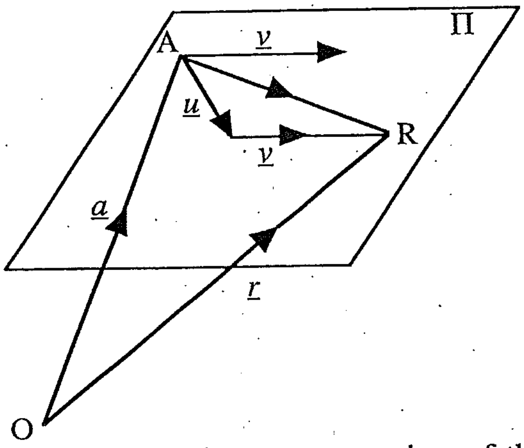
Let X with position vector $\underline{x}$ be a variable point.

Then the following statements are equivalent :
(i) X lies on Π. (ii) $\underline{n}$ is perpendicular to PX.
Hence (iii) $\underline{n}.(\underline{x}-\underline{p}) = 0$

i.e. $\underline{n}.\underline{x} - \underline{n}.\underline{p} = 0 \Rightarrow \boxed{\underline{n}.\underline{x} = \underline{n}.\underline{p}}$

This equation is true if and only if X lies on Π.
Since $\underline{n}$ and $\underline{p}$ are fixed, $\underline{n}.\underline{p} = k$ where k is a constant.
Therefore the equation of a plane can be written in the form $\underline{n}.\underline{x} = k$ where k is a constant, $\underline{n}$ is the normal vector to the plane and $\underline{x}$ is the position vector of any point on the plane.

(ii) in Parametric Form for the Vector Equation.



Consider the plane Π which is parallel to vectors $\underline{u}$ and $\underline{v}$ (where $\underline{u}$ is not parallel to $\underline{v}$) and which also contains the point A whose position vector is $\underline{a}$.

Let R be any point on Π with position vector $\underline{r}$ and so $\overrightarrow{AR}$ lies on Π.

If R is any point on this plane, $\overrightarrow{AR} = \lambda\underline{u}+\mu\underline{v}$ where λ and μ are parameters.

If $\underline{r}$ is the position vector of R, $\underline{r} = \underline{a} + \overrightarrow{AR}$

i.e. $\boxed{\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}}$

Thus any equation of the form $\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}$, where λ and μ are parameters, represents the plane parallel to the vectors $\underline{u}$ and $\underline{v}$ and containing the point $\underline{a}$.

(iii) in Symmetrical or Cartesian form.

In coordinate terms, if $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$ and $\underline{x} = x\underline{i} + y\underline{j} + z\underline{k}$ then the equation of the plane, with the aid of the scalar product is written

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = k \quad \text{or} \quad ax + by + cz = k$$

Note $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is called the normal vector

or its components are called the **direction ratios**

or if equal to the unit vector, **direction cosines.**

Examples

1. Find a parametric equation of the plane containing the three points A, B and C whose coordinates are (2, 1, 3), (7, 2, 3) and (5, 3, 5) respectively.

The position vectors of A, B and C are:-

$\underline{a} = 2\underline{i} + \underline{j} + 3\underline{k}$, $\underline{b} = 7\underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{c} = 5\underline{i} + 3\underline{j} + 5\underline{k}$

$\overrightarrow{AB} = (7\underline{i} + 2\underline{j} + 3\underline{k}) - (2\underline{i} + \underline{j} + 3\underline{k}) = 5\underline{i} + \underline{j}$

$\overrightarrow{AC} = (5\underline{i} + 3\underline{j} + 5\underline{k}) - (2\underline{i} + \underline{j} + 3\underline{k}) = 3\underline{i} + 2\underline{j} + 2\underline{k}$

The parametric equation is therefore

$\underline{r} = 2\underline{i} + \underline{j} + 3\underline{k} + \lambda(5\underline{i} + \underline{j}) + \mu(3\underline{i} + 2\underline{j} + 2\underline{k})$

2. Find the Cartesian equation of the plane containing the points A(0, 1, – 1), B(1, 1, 0) and C(1, 2, 0).

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix},\ \overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-1\\0\\1\end{pmatrix} \text{ from } \begin{matrix}\overrightarrow{AB} & 0 & 1 & 1 & 0\\ \overrightarrow{AC} & 1 & 1 & 1 & 1\end{matrix}$$

Hence the normal vector has components $\begin{pmatrix}-1\\0\\1\end{pmatrix}$

The equation of the plane is of the form $-x + z = k$
Since A(0, 1, – 1) lies on the plane, substitute in the above equation to find k. i.e. $k = -1$
=> The equation of the plane is $-x + z = -1$ or $x - z = 1$

3. Find the equation of the plane containing the points A(1, 2, 3), B(0, 3, 2) and C(3, 0, 5).

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}0\\3\\2\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix},\ \overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix}3\\0\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\-2\\2\end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}0\\0\\0\end{pmatrix} \text{ from } \begin{matrix}\overrightarrow{AB} & 1 & -1 & -1 & -1\\ \overrightarrow{AC} & -2 & 2 & 2 & -2\end{matrix}$$

The **normal** vector has components $\begin{pmatrix}0\\0\\0\end{pmatrix}$ which is impossible

Therefore the points A, B, C must be collinear and an infinity of planes must pass through A, B and C.

Notice that $\overrightarrow{AB} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}2\\-2\\2\end{pmatrix} = 2\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

4. Find the Cartesian equation of the plane through (–1, 2, 3) containing the direction vectors $8\underline{i} + 5\underline{j} + \underline{k}$ and $-4\underline{i} + 5\underline{j} + 7\underline{k}$.

The normal vector $\underline{n}$ is perpendicular to both the given vectors.

Let $\underline{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ and so $\begin{pmatrix}a\\b\\c\end{pmatrix}.\begin{pmatrix}8\\5\\1\end{pmatrix} = 0$ and $\begin{pmatrix}a\\b\\c\end{pmatrix}.\begin{pmatrix}-4\\5\\7\end{pmatrix} = 0$

i.e. $8a + 5b + c = 0$ and $-4a + 5b + 7c = 0$
Solve simultaneously to find b and c in terms of a, we obtain $b = -2a$ and $c = 2a$.

i.e. $\underline{n} = \begin{pmatrix}a\\2a\\-2a\end{pmatrix}$ or $\begin{pmatrix}1\\2\\-2\end{pmatrix}$

The equation of the plane is therefore in the form

$$x + 2y - 2z = k$$

Since A(–1, 2, 3) lies on the plane, substitute in the above equation to find k. i.e. $k = -3$

The equation of the plane is $x + 2y - 2z = -3$

5. Find the Cartesian equation of the plane containing the point P(3, –2, –7) and the line $\frac{x-5}{3} = \frac{y}{1} = \frac{z+6}{4}$.

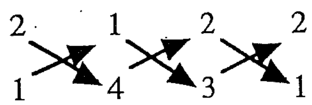
The point Q(5, 0, –6) lies on the line and the point P(3, –2, –7) lies on the plane.

A direction vector on the plane $\overrightarrow{PQ} = \begin{pmatrix}5\\0\\6\end{pmatrix} - \begin{pmatrix}3\\-2\\-7\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix}$

The direction vector of the line $= \begin{pmatrix}3\\1\\4\end{pmatrix}$

The normal vector to the plane is

$\begin{pmatrix}2\\2\\1\end{pmatrix} \times \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}7\\-5\\-4\end{pmatrix}$ from



The equation of the plane is therefore in the form

$$7x - 5y - 4z = k$$

Since P(3, –2, –7) lies on the plane, substitute in the above equation to find k. i.e. $k = 59$

The equation of the plane is $7x - 5y - 4z = 59$

Exercise 4

1. Find the Cartesian equation of the plane:-

(a) with norm $\begin{pmatrix}3\\-2\\7\end{pmatrix}$ and through the point (3, –1, 1)

(b) with norm $2\underline{i} - \underline{j} - \underline{k}$ and through (4, 1, –2)

2. Find the Cartesian equation of the plane through the given point, containing the stated directions :

(a) (1, 2, –3), $\begin{pmatrix}4\\-2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$

(b) (1, –1, 1), $3\underline{i} - 5\underline{j} - 7\underline{k}$, $-4\underline{i} - \underline{j} + 6\underline{k}$

3. Find the Cartesian equation of the plane through the following sets of three points :-

(a) ((2, 1, 3), (4, 1, 4), (2, 3, 6)

(b) (3, 0, 0), (0, 5, 0),(0, 0, 7)

4. Find the Cartesian equation of the plane containing both the point and the line given :

(a) $(5, 8, -4), \ \frac{x}{-4} = \frac{y-5}{1} = \frac{z+1}{0}$

(b) $(2, -5, 3), \ \frac{x-1}{-3} = \frac{y+7}{5} = \frac{z-3}{2}$

5. A plane is parallel to both lines

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-1}{4} \text{ and } \frac{x+1}{-1} = \frac{y}{2} = \frac{z}{1}$$

and passes through the point (1, 0, –1). Find its equation.

6. A plane passes through the points (0, 1, 2) and (1, –1, 0) and is parallel to the direction $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Find its equation.

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough). No reference

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 429 Exercise 17D Questions 1 – 12

Angles between Two lines, Two Planes, or a Line and Plane

(i) The angle between Two Lines.

The angle between two lines is the angle between their direction vectors and can be found using the Scalar Product.

Example Find the size of the angle between the lines $x - 1 = y = z - 1$ and $x = 1 + t$, $y = 5t$, $z = -t$.

The lines can be expressed in the symmetrical form

i.e. $\frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{1}$ and $\frac{x-1}{1} = \frac{y}{5} = \frac{z}{-1}$

Their direction vectors are $\underline{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

Using $\cos\theta° = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{5}{\sqrt{3}.3\sqrt{3}} = \frac{5}{9} \Rightarrow \theta = 56 \cdot 3$

(ii) The angle between Two Planes:-

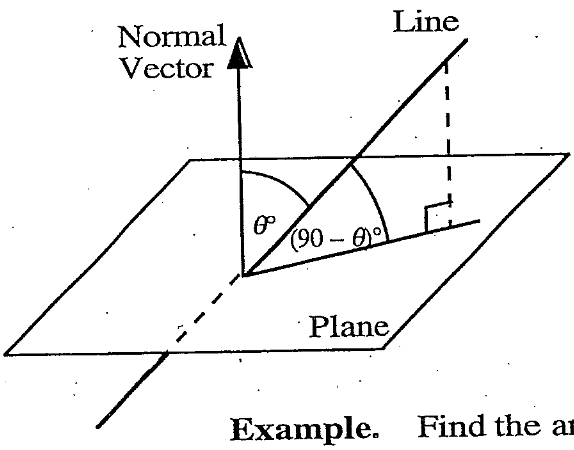
The angle between two planes is defined to be the angle between their <u>normal vectors</u>.

Example. Find the angle between the planes $x + 2y + z = 0$ and $x + y = 0$.

Their normal vectors are $\underline{a} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\underline{b} = \begin{pmatrix}1\\1\\0\end{pmatrix}$.

$$\cos\theta = \frac{\underline{a}\,.\,\underline{b}}{|\,\underline{a}\,||\,\underline{b}\,|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \pi/6$$

(iii) The angle between a Line and a Plane:-



If $\theta°$ is the angle between a line and the normal vector to the plane, then $(90 - \theta)°$ is the angle between the line and the plane.

Note (i) $(90 - \theta)°$ is the angle between the line and its projection on the plane.

(ii) $(90 - \theta)°$ is the smallest angle between the line and the plane.

Example. Find the angle between the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{0}$ and the plane $x + z = 0$

The normal vector of the plane $= \begin{pmatrix}1\\0\\1\end{pmatrix}$

The direction vector of the line $= \begin{pmatrix}1\\1\\0\end{pmatrix}$

By the scalar product

$$\cos\theta° = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = 60$$

The angle between the line and plane $= (90 - \theta)° = 30°$

The Intersection of Two Lines, Two or Three Planes, or a Line and Plane.

(i) The Intersection of Two Lines.

Two distinct lines in a plane are either parallel or intersecting. In three dimensions there are three possibilities : they may be parallel, or intersecting or skew (neither parallel nor intersecting).

The following example demonstrates how to find whether two lines intersect and if they do, how to find the point of intersection.

Example

$$\frac{x+9}{4}=\frac{y+5}{1}=\frac{z+1}{-2} \quad ..(1) \qquad \frac{x-8}{-5}=\frac{y-2}{-4}=\frac{z-5}{8} \quad ..(2)$$

$$\frac{x-8}{-5}=\frac{y-2}{-4}=\frac{z+15}{8} \quad ..(3)$$

(a) Find where lines (1) and (2) intersect.
(b) Find where lines (1) and (3) intersect.

=> (a) If there is a point (p, q, r) lying on both lines,

then $\frac{p+9}{4}=\frac{q+5}{1}=\frac{r+1}{-2}=\lambda$ and $\frac{p-8}{-5}=\frac{q-2}{-4}=\frac{r-5}{8}=\mu$

Therefore $p = 4\lambda - 9 = -5\mu + 8$ ......(4)
$q = \lambda - 5 = -4\mu + 2$ ......(5)
$r = -2\lambda - 1 = 8\mu + 5$ ......(6)

Solving (4) and (5) => $\lambda = 3$ and $\mu = 1$.

Substitute these values in (6) gives $-7 = 13$.

Since these values do not satisfy (6), we conclude that the lines (1) and (2) do not intersect.

(b) If there is a point (p, q, r) lying on both lines,

then $\frac{p+9}{4}=\frac{q+5}{1}=\frac{r+1}{-2}=\lambda$ and $\frac{p-8}{-5}=\frac{q-2}{-4}=\frac{r+15}{8}=\mu$

Therefore $p = 4\lambda - 9 = -5\mu + 8$ ......(7)
$q = \lambda - 5 = -4\mu + 2$ ......(8)
$r = -2\lambda - 1 = 8\mu - 15$ .....(9)

Solving (7) and (8) => $\lambda = 3$ and $\mu = 1$.

Substitute these values in (9) gives $-7 = -7$.

Since these values satisfy (9), we conclude that the lines (1) and (3) intersect.

Using $\lambda = 3$ (or $\mu = 1$) in (7), (8) and (9),

=> $p = 3$, $q = -2$ and $r = -7$.

Therefore lines (1) and (3) intersect at (3, –2, –7).

(ii) The Intersection of Two Planes.

Two non-parallel planes will always meet in a straight line.
Given the equation of two planes, we can proceed as follows:-

Example Find the equation of the line of intersection of the planes $3x - 5y + z = 8$ and $2x - 3y + z = 3$.

Method 1.

For any point (x, y, z) which lies on both planes, the values of x, y and z fit both equations simultaneously. Hence eliminating z from both equations (by subtraction in this case) gives $x - 2y = 5$.

There are infinitely many pairs of values of x and y which satisfy this equation, but if we choose a value of x then the value of y is fixed and vice-versa.

Let $y = t$, then $x = 5 + 2t$ and substituting these expressions for x and y into $3x - 5y + z = 8$ gives

$$3(5 + 2t) - 5t + z = 8 \quad \Rightarrow \quad z = -7 - t$$

i.e. $x = 5 + 2t$, $y = t$, $z = -7 - t$ are the parametric equations of the line

or in Cartesian form $\frac{x-5}{2} = \frac{y}{1} = \frac{z+7}{-1}$ $(= t)$

Method 2.

To find the equation of a line, we require its direction vector and a point on the line.
The direction vector of the line is perpendicular to both normal vectors of the planes.

The normal vectors of the planes are $\underline{a} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$:

The direction vector of the line of intersection must be parallel to $\underline{a} \times \underline{b}$.

$$\underline{a} \times \underline{b} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

Let $z = 0$, hence $3x - 5y = 8$, $2x - 3y = 3$.
Solving these gives $x = -9$, $y = -7$.
i.e. $(-9, -7, 0)$ is a point on the line.

The equation of the line in Cartesian form is

$$\frac{x+9}{-2} = \frac{y+7}{-1} = \frac{z}{1} \quad \text{or} \quad \frac{x+9}{2} = \frac{y+7}{1} = \frac{z}{-1}$$

which can be turned into $\frac{x-5}{2} = \frac{y}{1} = \frac{z+7}{-1}$

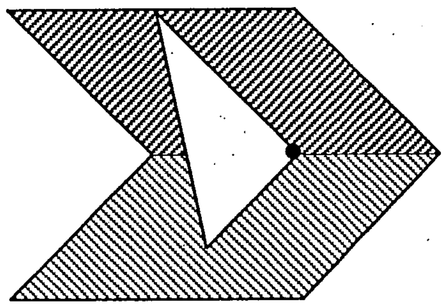
by subtracting 7 from each part.

(iii) The Intersection of Three Planes.

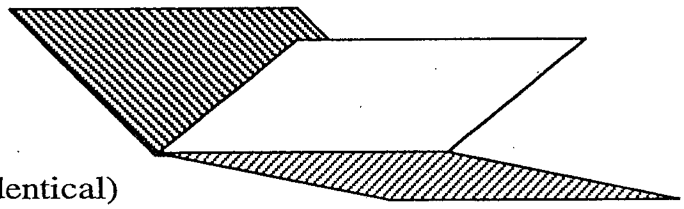
The solution of a system of 3 planes is a point common to the 3 planes. If the matrix of the coefficients is singular (see Matrices) then the equations do not have a unique solution.

The equations could either have:-

(a) a unique solution, intersecting at one point.



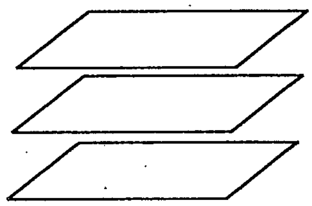
(b) a linear solution, in which case one equation is a linear combination of the other two.



(or the 3 planes are identical)

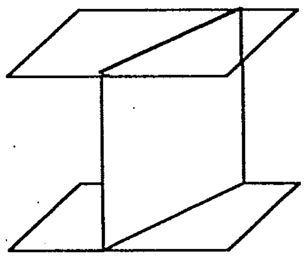
There are an infinite set of points which are common to all 3 planes.

(c) no solution, in which cases the three planes are parallel, →



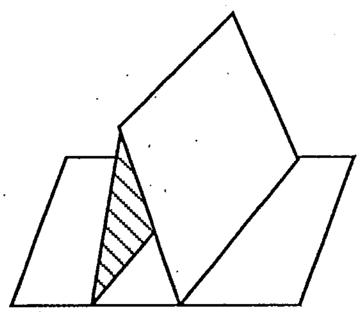
or

two are parallel →



or

one plane is parallel to the line of intersection of the other two. →



Example

Show that the planes:-

A : $2x - y + 5z = -4$
B : $3x - y + 2z = -1$
C : $4x - y - z = 2$

intersect on a line and find its equation.

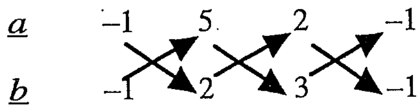
The normal vector of A : $\underline{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$

The normal vector of B : $\underline{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

The normal vector of C : $\underline{c} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

The directon vector of the line of intersection of A and B is parallel to $\underline{a} \times \underline{b}$.

i.e. $\underline{a} \times \underline{b} = \begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix}$ from



If $(\underline{a} \times \underline{b}).\underline{c} = 0$, the direction vector of the line of intersection is perpendicular to the normal vector of plane C.

=> the line of intersection of planes A and B must be parallel to the plane C.
i.e. the three planes intersect on a line.

Let $z = 0$, A : $2x - y = -4$
B : $3x - y = -1$
C : $4x - y = 2$

From A and B, $x = 3$ and $y = 10$.
A point on the line of intersection of A and B is therefore (3,10, 0).

This point also satisfies C, since $12 - 10 - 0 = 2$.
i.e. (3,10, 0) lies on all three planes.

The equation of the line is $\frac{x-3}{3} = \frac{y-10}{11} = \frac{z}{1}$

Note **If the planes intersect at a point, the soltion can be found by Gaussian elimination or Matrices (next outcome).**

Exercise 5 1. Calculate the acute angle between the following pairs of lines :

(a) $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z}{2}$, $\frac{x+3}{3}=\frac{y+1}{0}=\frac{z-2}{-4}$

(b) $x-1=y=z-1$, $x=1+t,\ y=5t,\ z=-t$

(c) $\frac{x+4}{3}=\frac{y-1}{5}=\frac{z+3}{4}$, $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$

(d) $\underline{r}=4\underline{i}-\underline{j}+\lambda(\underline{i}+2\underline{j}-2\underline{k})$, $\underline{r}=\underline{i}-\underline{j}+2\underline{k}-\mu(2\underline{i}+4\underline{j}-4\underline{k})$

2. Find the acute angle between the following pairs of planes :

(a) $2x+2y-3z=3$, $x+3y-4z=6$

(b) $5x-14y+2z=13$, $6x+7y+6z=-23$

(c) $\underline{r}.(\underline{i}-\underline{j})=4$, $\underline{r}.(\underline{j}+\underline{k})=1$

(d) $\underline{r}.(\underline{i}+\underline{j}+\underline{k})=1$, $\underline{r}.(\underline{i}-\underline{j}+\underline{k})=0$

3. Find the acute angle between the following lines and planes :

(a) $\frac{x}{4}=\frac{y-1}{-1}=\frac{z+3}{-5}$, $x-2y+4z=-3$

(b) $\frac{x-2}{2}=\frac{y+1}{6}=\frac{z+3}{3}$, $2x-y-2z=4$

(c) $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$, $10x+2y-11z=3$

(d) $\underline{r}=\underline{i}-\underline{j}+\lambda(\underline{i}+\underline{j}+\underline{k})$, $\underline{r}.(\underline{i}-2\underline{j}+2\underline{k})=4$

4. Find the coordinates of the point of intersection of these lines :

(a) $\frac{x-4}{1}=\frac{y-8}{2}=\frac{z-3}{1}$, $\frac{x-7}{6}=\frac{y-6}{4}=\frac{z-5}{5}$

(b) $\frac{x-2}{1}=\frac{y-9}{2}=\frac{z-13}{3}$, $\frac{x+3}{-1}=\frac{y-7}{2}=\frac{z+2}{-3}$

5. Find the equation of the line of intersection of the following two planes :

(a) $x+y+2z=2$, $x-y-z=5$

(b) $2x-y=3$, $x+y+4z=1$

(c) $2x+3y+z=8$, $x+y+z=10$, $3x+5y+z=6$.

6. Find the coordinates of the point of intersection of the following line and plane :

(a) $x=4+t,\ y=1-t,\ z=3t$, $2x+4y+z=9$

(b) $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-1}{4}$, $x-2y+3z=26$

7. A plane contains the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-1}{2}$ and the plane is parallel to the line $\frac{x}{1}=\frac{y+7}{2}=\frac{z}{3}$.

   Find the equation of the plane.

8. (a) Find the coordinates of the point A in which the line L with equation $\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z+3}{2}$ meets the plane Π with equation $3x-y+z=10$.

   (b) Hence find the equation in cartesian form for the line through A lying wholly in the plane and perpendicular to the line.

9. (a) Find parametric equations for the line L joining the points A(2, –4, 3) and B(4, 0 –5).

   (b) Verify that L is perpendicular to the line M joining the point A to C(–2, –6, 1).

   (c) Determine the equation of the plane containing L and M.

10. (a) Show that the line L with parametric equations $x=2-t$, $y=-3+2t$, $z=-1-4t$ lies on the plane Π with equation $2x+3y+z=-6$.

    (b) Find parametric equations for the line M through the point (3, 2, –4) perpendicular to Π.

    (c) Prove that M meets Π at a point lying on L.

11. (a) Show that the line L joining the points A(2, 1, –1) and B(3, –2, 1) is perpendicular to the line M with parametric equations $x=11+4t$, $y=3+2t$, $z=1+t$.

    (b) Find the equation of the plane Π through L perpendicular to M and prove that Π meets M at a point C equidistant from A and B

12. (a) Find paramtetric equations for the line joining A(1, –1, 2) and B(4, 5, –7).

    (b) Prove that AB intersects the line with parametric equations $x=6+4t$, $y=2+t$, $z=1+2t$ at right angles.

    (c) Find the coordinates of the point of intersection of lines.

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

No reference

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 420 Exercise 17C Questions 13 – 18
Page 429 Exercise 17D Questions 16 – 28

Answers

Exercise 1

1.(a) Number, 8 (b) Vector, $5\underline{i} + 7\underline{j} + 9\underline{k}$ (c) Vector, $4\underline{i} + 4\underline{k}$ (d) Number, 45.

2. $AB = \sqrt{38}$, $BC = 3\sqrt{6}$, $CA = \sqrt{10}$,
$\cos A = -\dfrac{3}{2\sqrt{95}}$, $\cos B = \dfrac{41}{6\sqrt{57}}$, $\cos C = \dfrac{13}{6\sqrt{15}}$, Area $= \dfrac{1}{2}\sqrt{371}$

3. Proof

4. $\dfrac{1}{\sqrt{2}}\underline{i} + \dfrac{1}{\sqrt{2}}\underline{j}$, $\dfrac{1}{\sqrt{2}}\underline{i} - \dfrac{1}{\sqrt{2}}\underline{j}$.

5. (a) $\dfrac{-3+2\sqrt{2}}{6}\underline{i} + \dfrac{\sqrt{2}}{6}\underline{j} + \dfrac{3+2\sqrt{2}}{6}\underline{k}$, $\dfrac{-3-2\sqrt{2}}{6}\underline{i} - \dfrac{\sqrt{2}}{6}\underline{j} + \dfrac{3-2\sqrt{2}}{6}\underline{k}$
(b) Proof

Exercise 2

1.(a) $-10\underline{i} + 7\underline{j} - 16\underline{k}$ (b) $5\underline{i} - 5\underline{k}$ (c) -20 (d) -15 (e) -125

2.(a) $-11\underline{i} + 14\underline{j} + \underline{k}$ (b) 2 (c) -2 3. Proof

4. $5\underline{i} + 31\underline{j} + 14\underline{k}$ 5. (a) $-3\underline{j} - 3\underline{k}$ (b) $-3\underline{j} - 3\underline{k}$ (c) Proof

6. $-\dfrac{1}{13}(3\underline{i}+4\underline{j}+12\underline{k})$, $\dfrac{1}{13}(3\underline{i}+4\underline{j}+12\underline{k})$ 7. $11\frac{1}{2}$ 8. Proof

Exercise 3

1. $\dfrac{x-5}{3} = \dfrac{y+2}{1} = \dfrac{z-6}{4}$, $\underline{x} = \begin{pmatrix}5\\-2\\6\end{pmatrix} + t\begin{pmatrix}3\\1\\4\end{pmatrix}$

2. $\dfrac{x-2}{7} = \dfrac{y+3}{3} = \dfrac{z+7}{-2}$, $\underline{x} = \begin{pmatrix}2\\-3\\-7\end{pmatrix} + t\begin{pmatrix}7\\3\\-2\end{pmatrix}$

3.(a) $\dfrac{x-3}{2} = \dfrac{y-2}{-15} = \dfrac{z+7}{3}$ (b) $\dfrac{x-12}{2} = \dfrac{y-7}{2} = \dfrac{z-1}{1}$

(c) $\dfrac{x-3}{3} = \dfrac{y}{0} = \dfrac{z}{5}$ (d) $\dfrac{x}{-5} = \dfrac{y}{2} = \dfrac{z}{-3}$

Exercise 4 Pages

1. (a) $3x - 2y + 7z = 18$ (b) $2x - y - z = 9$
2. (a) $x - y - 2z = 5$ (b) $x + 2y + z = 0$
3. (a) $x + 3y - 2z = -1$ (b) $35x + 21y + 15z = 105$
4. (a) $3x + 12y + 17z = 43$ (b) $4x - 2y + 11z = 51$
5. $5x + 6y - 7z = 12$
6. $4x + 3y - z = 1$

cont'd .....

Exercise 5

1. (a) 70·5° (b) 56·3° (c) 22·5° (d) 0
2. (a) 25·2° (b) 70·2° (c) 60° (d) 70·5°
3. (a) 28·5° (b) 22·4° (c) 22·4° (d) 11·1°
4. (a) (1, 2, 0) (b) (–1, 3, 4)
5. (a) $\frac{x-3}{15}=\frac{y-1}{27}=\frac{z}{7}$ (b) $\frac{x+1}{5}=\frac{y+1}{23}=\frac{z+1}{22}$

   (c) $\frac{x-22}{2}=\frac{y+12}{-1}=\frac{z}{-1}$
6. (a) (1, 4, –9) (b) (7, 4, 9)
7. $7x + y - 3z = 6$
8. (a) (3, 0, 1) (b) $\frac{x-3}{1}=\frac{y}{4}=\frac{z-1}{1}$
9. (a) $x = 2 + t, y = -4 + 2t, z = 3 - 4t$ (b) Proof

   (c) $2x - 3y - z = 13$
10. (a) Proof (b) $x = 3 + 2t, y = 2 + 3t, z = -4 + t$ (c) Proof
11. (a) Proof (b) $4x + 2y + z = 9$, C(3, –1, –1)
12. (a) $x = 1 + t, y = -1 + 2t, z = 2 - 3t$ (b) Proof

    (c) (2, 1, –1)